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Mathematics in High School and College

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The publication of a high-school mathematics text, *Senior Mathematics*, by Douglass and Kinney (Henry Holt and Co., 1945) furnishes me an opportunity to express some opinions concerning the relation between mathematics in the secondary school and in the college.

The senior author is Harl Douglass, a friend and respected colleague, Head of the College of Education at the University of Colorado, and an expert in the field of secondary education and relations between secondary schools and colleges. Precisely because I have always found him a broad-minded liberal individual, the basic tendencies and claims of Douglass' new book seem to me particularly disturbing.

The following comments are not intended as criticism or review, but rather are offered in a spirit of bewilderment at the ever widening differences of opinion between opposing groups in education.

The preface of the book may be taken reasonably as a reflection of the author's philosophy of education, and it will simplify matters if I group my comments around sentences taken from the preface which can in this case be done without distorting the meaning in lifting the phrases out of their setting. (Italics mine in all quotations from the book.)

The first sentence of the preface is: *Senior mathematics is an integrated selection of procedures for solving the more common mathematical problems of shop, business, military life, travel, science, home, health, and safety.*

This sentence states the aims of the book with perfect clearness. The mathematics is to be developed as a tool subject, the student is to be taught to solve problems which he is likely to encounter in daily life. Here it is possible for me to say that this program, taken literally, is carried out expertly and that the range of topics covered is very impressive. Thus, in chapter VIII, of about 50 pages, under the general title, *Fractions and Percentage in Home Accounting*, the following matters are taken up and are dealt with in a manner which should be really helpful to the average housewife or home owner. Indeed, this chapter taken by itself would seem to me of great value as a pamphlet intended for adult education, if one is willing to consider solely the ability to obtain numerical results mechanically.

The topics are, with some sub-headings omitted: *Family Income Accounts—The Cash Account—The Cost of Owning and Operating a Home—Computing Taxes on Your Income—Depreciation—Percentage and Buying Insurance—Financing the Cost of the Home—Providing for the Future (Computing Interest on Savings. Using Tables to find Compound Interest. Calculating Interest Compounded Semi-annually or Quarterly)—Providing for the Future of*

your Dependents—Investing your Money (Investing in Government Bonds. Government Bonds. Bonds Issued by Business Corporations. Purchasing Bonds. Investing in Stocks. Computing Annual Return. Investing in Real Estate.)

The question of the importance of all these matters, the desirability of their inclusion in the high school curriculum, is not under discussion. One can only greet with warm approval the topics contained in this chapter.

The objections I must raise are based on the fact that, true to the program contained in the opening sentence of the preface quoted above, all emphasis is placed on achieving ability to solve problems, with a very minimum of emphasis on developing understanding of the underlying mathematical principles and the theorems involved.

To illustrate my point: it is obvious that the main mathematical tool for dealing with the important problems just mentioned is the theory of geometrical progressions. In this book of 432 pages the whole theory of geometrical progressions is contained on less than one page of text, which I cannot refrain from copying verbatim.

**Geometric Progression*[1]

In a geometric progression each term has a constant ratio to the preceding term. For example in the geometric ratio 1, 3, 9, 27, 81 the ratio is 3 to 1, or simply 3. Below are some geometric progressions. What is the ratio in each progression?

1. 8, 32, 128. *4.* $\frac{1}{4}$, $\frac{1}{2}$, 1.
2. .5, 2.5, 12.5. *5.* x, x^2, x^3.
3. 12, 6, 3. *6.* $a+b$, $2a+2b$, $4a+4b$.

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To find a certain term in a geometric ratio the following formula is used: $l=anr^{n-1}$ *in which l is the term desired, a is the first term, r is the ratio, and n is the number of terms desired.*

For example: What is the sixth term in the series 4, 8, 16, 32? $l=4r^{n-1}$ *or* $4r^5$ *which is* $4(2)^5$ *or* 128.

Find:

1. *The fifth, sixth and seventh terms of the progression:* 1, 3, 9
2. *The sixth, seventh and eighth terms of the progression:* 1, 2, 4
3. *The fifth, seventh and ninth terms of the progression:* 1, .1, .01
4. *The fifth, eighth and tenth terms of the progression:* $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$
*5. *The sixth, eighth and ninth terms of the progression:* 2, −4, 8, −16
*6. *The fifth, seventh and ninth terms of the progression:* $\sqrt{3}$, 3, $3\sqrt{3}$, 9

**To find what term a certain number or quantity is in a known progression, you use the formula*

$$n=\frac{\log l-\log a+\log r}{\log r}.$$

Example: What term is 1,000,000 in the progression 1, 10, 100?

Solution: $\log l=6$, $\log a=0$, $\log r=1$

$$n=\frac{6-0+1}{1}=7.$$

Hence 1,000,000 is the seventh term of the progression.

This is followed by a list of ten starred problems, from which I copy the first two and last two.

**Exercises*

1. *What term is 2187 of the progression* $\frac{1}{9}$, $\frac{1}{3}$, 1?
2. *What term is .0000001 of the progression* 10, 1, .1?
9. *The fifth term of a series is 4 and the eighth term is 32. What is the ratio?*
10. *If a man could double his money every 2 months, how long would it take him, starting with \$2 to accumulate \$512?*

To justify the character of some of the remarks I shall make it must be understood that the passage quoted (pp. 156–7)

[1] The * is explained in the preface as follows: "For more advanced or more industrious students, it supplies an abundance of supplementary materials (starred)—more abstract and difficult problems and exercises, challenging to the rapid worker and demanding a high degree of originality in attack."

contains absolutely everything the book gives on the theory of geometric progressions. The sum formula is not mentioned. In chapter VIII compound interest problems involving a small number of terms are worked out by a step-by-step process, as:

On Jan. 1, 1943 H. W. had $500 in a saving bank at 2% interest, compounded semiannually. What will be the amount to his credit at the end of a 2 yr. period? The calculation is:

Principal	*Jan. 1, 1943*	$500
Interest	*July 1, 1943*	5
Principal	*July 1, 1943*	505
Interest	*Jan. 1, 1944*	5.05
Principal	*Jan. 1, 1944*	510.05
Interest	*July 1, 1944*	5.10
Principal	*July 1, 1944*	515.15
Interest	*Jan. 1, 1945*	5.15
		520.30

For problems involving longer periods a short compound interest table is furnished. For example: *$300—for 12 years at $3\frac{1}{2}$%, compounded annually*—the whole work consists in looking up the entry for $1 capital in the table, and multiplying by 300. No explanation of the construction of the table is given.

Infinite geometric progressions are not mentioned in the book; non-terminating decimals for ordinary fractions are disposed of by such problems as: *Change 4/9 and 3/17 to decimals to the nearest thousandth.*

While infinite geometric progressions are not usually supposed to be taken up in first year high school algebra, it remains disturbing that as a consequence students can have no clear comprehension of so basic a concept as the repeating decimal. The bright youngster of whom the story is told that his teacher sent him to the board to change $\frac{1}{3}$ to a decimal fraction, and who during the noon recess covered all boards with his continued divisions, and then greeted his teacher with the words, "It's got to end sometime, and I want to see how it ends," would not receive much enlightenment in a current introductory high school algebra course.

Factoring is not considered at all, not even factoring of a^2-b^2. Consequently, the solution of quadratic equations by factoring is omitted. In fact, the theory of quadratic equations $at^2+bt+c=0$ is completely dealt with in one paragraph (p. 126): *To find t you make use of the following formula, which is known as the Quadratic Formula and is used to solve equations of this type for the unknown quantity:*

$$t=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

In this formula a is the coefficient of the squared term, b is the coefficient of the other letter term, c is the third term.

No attempt at derivation of the "Quadratic Formula," no hint that the equation may not have real roots!

Negative numbers are introduced in the following manner (p. 109):

**Calculating with Negative Quantities*

Numbers can be either negative or positive. Positive numbers are those which are greater than zero. Negative numbers are less than zero. When the temperature is 10° below zero, it is −10°, and that is a negative number. . . .

The whole book deals essentially with numerical problems and, in view of the incredibly poor preparation of most high school students entering the university, this aspect can hardly be over-emphasized. (At the University of Colorado, where we have a fairly wide distribution of students from all parts of the country, we have valid statistical evidence that the mathematical preparation of high school students, and in arithmetic in particular, has deteriorated steadily during the last years, and that the expected re-invigoration due to emphasis on mathematics for war demands has not

in any way materialized. This is not the occasion to discuss this phenomenon and its causes—and predictable effects.)

Formulae from algebra, plane and solid geometry are given and used without derivation except for trivial cases. The theorem of Pythagoras is simply stated: *The square of the hypotenuse is equal to the sum of the squares of the other two sides, or*

$$c^2=a^2+b^2$$

when a and b are the sides and c the hypotenuse, and is then applied to practical problems.

I believe that I have said enough to give a fair description of the type of book I am discussing, and I have gone to such detail because this text seems to me a perfect illustration of the divergent tendencies in the secondary schools as compared with colleges and universities.

For the sake of argument let me assume, although it has never been proven, that the American ideal of high school education for the whole population implies a screwing down of levels and goals, and that this might mean that the idea of mathematics in any sense but a pure tool-subject may have to be discarded for the great majority of students. It would be a sad comment on the intellectual level of the nation, but it then would be possible that even the best methods of teaching may not be able to impart mathematics to any but a relatively small group except by memorizing rules, applying them mechanically to a large variety of typical problems, and drilling and drilling and drilling until the work becomes second nature. The tendency is strongly in this direction at the present time, and Douglass' book seems to me a perfect exponent of these aims.

However, let me return to the "Preface," and see what claims are made for the book beyond the immediate tool value: *But sequence, system, and logic have not been sacrificed to socialization.* As far as logic is concerned, the claim is founded to the extent that no actual logical errors were noticed (however, division by zero is not explicitly excluded, p. 105); but can one honestly say that logic is not sacrificed when the part where logic really enters into mathematics, the derivation of the formulae, has been almost completely thrown overboard? And "sequence" and "system"? It is admittedly difficult to find a comprehensive unifying principle in elementary algebra, while it is simple in geometry and trigonometry. But unless in some manner the structure of a mathematics course is made to stand out clearly, there is no "system," and the "sequence" degenerates into a succession of unrelated topics.

Students who will have worked conscientiously through the book may have acquired the technical and mechanical ability to solve "the more common mathematical problems of shop, business, military life, travel, science, home, health, and safety," although it does look like a large bill to fill, but they will not have gained any understanding of the "sequence, system, and logic" of any branch of mathematics.

The absence of structural mathematical unity and the lack of logical development of the individual theorems leave us truly naught but the shadow of the ghost of mathematics.

These objections must be contrasted with the further claim of the preface: *Explanations are clear and as non-technical as possible, while mathematically sound.*

Where does the interest of the college and university enter into this discussion? After all, it might be argued—short-sighted though it would ultimately turn out to be—that the *Senior Mathematics* is intended exclusively as a course for students who will not continue their education after high school. Short-sighted, because these students represent the vast majority of all students, and it is absurd to imagine that all who do not go to college are intellectually incapable of digesting even a minimum of pure logic. But it becomes impossible for the college to ignore this book in view of the further

statement of the preface: *A student who masters this book will have as preparation for college study the equivalent of two years of high school mathematics, including the procedures basic to college courses in science and mathematics and business.* This sentence forms the real basis for my discussion of Douglass' book. By failure to protest in the most definite form, colleges and universities will soon discover that they have by their silence acquiesced in, and accepted, the claim of the author that his book satisfies the entrance requirements of institutions demanding a year of high school algebra and a year of high school geometry. Colleges may then frankly admit complete defeat in their long struggle to retain at least a minimum of sound mathematical knowledge and training for high school students who will take courses in college mathematics, and resign themselves to teaching the complete contents of the present high school algebra and geometry courses; for, and make no mistake about it, all but a negligible amount of the mathematics contained in *Senior Mathematics* will have to be taught over again in college, since a student who knows only this text has had no opportunity to learn what constitutes a mathematical proof, or to develop any feeling for the necessity of a logical proof.

Up to the present time the colleges have labored under inadequate preparation of high school students for Freshman college courses. However, the theoretical content of the high school courses has, on the whole—leaving aside some aspects of the Progressive Education plans—emphasized the structure of the mathematics course as a logical unit, composed of separate parts studied by logical processes; and we have considered the poor results of high school courses as due to the discrepancy between desire and performance. But the whole basis is shifted entirely if we accept the program represented by this book. It is true that the trends have been in this direction for half a century. First the justified complaints against the exaggerated barren formal development which was not at all adapted to—or even originally intended for—the adolescent mind; then the emphasis on integrating into high school courses only so much as is of immediate practical importance; then the efforts to pick up this material piecemeal as it is required in applications to other fields; until we now face the full-grown development of these efforts: the nearly complete sacrifice of algebra and geometry as total logical structures, and abandonment of all serious attempts to prove results in detail.

Friends and colleagues, college professors and high school teachers, insist on reassuring me that I view the education situation in the country too darkly. (I should, however, add that on the whole classroom teachers seem much more inclined to share my misgivings than are my university friends.) Ultimately, I must trust my own experience and my own eyes. Thirty-five years ago the university Freshman courses in algebra were on a distinctly higher level than they are at present; students who did not have a fair grounding in arithmetic and a certain understanding of what properly belongs to elementary algebra were in a far smaller minority than they are today. We did not have to start our college algebra on so low a level, and we carried our students considerably farther, with much more emphasis on proofs than at present. The deterioration over a third of a century—the span of my own experience—has been steady, but progressing at so slow a rate that the effects of this creeping paralysis are perceptible only when we look over stretches of decades. Making exceptions to entrance requirements here and there, lowering them gradually, eliminating this and that topic from the college course without changing its title, in order to gain time for material not properly digested or neglected altogether in high school—such have been the easy steps of our descent to inferno. I can only invite doubting readers to compare text-books of four decades ago with

those used now, and to study in university catalogues the outlines of algebra courses given before the First World War for comparison.

I am stating facts, without expressing criticism of either secondary schools or colleges. Indeed, any blame belongs to all of us; at least as much to the colleges which train the teachers, as to the teachers; most of all, the blame belongs to our unclear ideas and diffuse philosophies of education. But the fact itself remains, as facts always do, with all of their implications and threats for the future!

May I present a few definite data?

Of 421 high school students from all parts of the country who entered the College of Arts and Sciences at the University of Colorado in October, 1944, nearly one-half (204) could not multiply 74.5 by 0. About thirty per cent (124) could not add $8\frac{2}{9}$ and $3\frac{1}{2}$. Of these, twenty-six did not attempt to answer this question at all; among replies given are 73; 3.50; 17.22; 1431.1; 38; $28\frac{7}{9}$; $93\frac{12}{18}$ $\frac{11}{18}$. Only thirty-nine of these poor students had even the integral part correct. Pressure of time could not possibly account for this poor showing, since the question is in the first third of one of the sections of the test.

Particularly startling are the results to the following question: *If c is the cost per yard, what is the cost per foot?* This "problem" was the very first one of a section of the test. 157 of the 421—more than one out of three—gave incorrect answers or omitted the question, although no choice was allowed. It may be interesting to see a list of answers offered (in parentheses the number of times each answer was given, if repeated): $3c$ (31); $c-3$ (10); $12c$ (5); $3/c$ (4); $c-2$ (4); $c-24$ (3); $c/2$ (2); $c+3$ (2); $c/12$ (2); $36c$ (2); $4c$ (2); $c/4$; $36/c$; $c/6$; $c/5$; $12/c$; $c/3$ or $c-3$; $4c$; $-3c$; $-12c$; $c/36$; $c+12$; $c-4$; c; $1/(3c)$.

But while these answers, absurd as they are, still show some glimmering of low-grade intelligence, what shall one say of the so-called mental processes which prompted the following replies? x (5); a (4); b (3); c^3 (3); c^2 (2); $c-x$ (2); d; $c-3A$; $F-c$; $3\sqrt{c}$; cx; $c-c^2$; $c=x-y$; c^{12}; $c=2x$; $c-3x$; c^6+c^6?

Thus about one-third of the entering students, who were given a clean bill of health by the secondary schools, as far as arithmetic, algebra and geometry are concerned, and whose parents live in the faith that their children have received a good and thorough basic education, cannot perform the simplest of arithmetical computations. They are thrown into the university, where they only serve to lower standards and decrease the efficiency of teaching, at the same time depriving able and alert students of the opportunities of instruction suited to their talents. Segregation of superior students has been tried often, but for various reasons fails to offer a satisfactory solution.

When we pass to what goes under the name of "algebra," we find that 378 of our 421 (90%) could not add $\frac{1}{x}+\frac{1}{y}$. This question was just in the middle of a section of the test, and time pressure could not count. Thus, nine out of ten students entering the College of Arts and Sciences do not know how to add the simplest literal fractions, after one or one and a half or two years of high school algebra. (An examination of the much smaller group of Engineering students (exclusive of Navy students, who did not take the test)) gives different results, as should be expected from the fact that these students are required to offer $1\frac{1}{2}$ years of high school algebra, 1 year plane geometry and $\frac{1}{2}$ year solid geometry. They represent, both by training and by interest, a highly selected group. Of 56, seven could not multiply 74.5 by 0; eleven added their $8\frac{2}{9}$ and $3\frac{1}{2}$ incorrectly, eight could not change the price per yard to price per foot. Obviously, the level of reasoning has been raised somewhat! However, the improvement does not go too far. Thirty-seven of the fifty-six (two-thirds) cannot add $\frac{1}{x}+\frac{1}{y}$.

As many give $\frac{1}{x+y}$ as give correct answers. And this after at least three years of high school mathematics!)[2]

In this connection I cannot refrain from recommending to my readers most sincerely the excellent article by F. D. Murnaghan, "The Teaching of Mathematics," in the Dec. 1, 1944 issue of *Science*.

In the criss-cross of currents sweeping our present day educational philosophies we must be prepared to have evidence of poor preparation of our university Freshmen turned against us in the argument that formal algebra and geometry should be abolished from the high school except for superior students, and relegated to the college and university. Although in fact much of the college teaching has to be handled at present as if the student knew no algebra or geometry, stabilization of the situation at this level would be a backward step of such decisive character that our "universities" would become the laughing stock of the civilized world.

The argument that our figures indicate that algebra and geometry are above high school intellectual level is refuted by the fact that most of our well prepared students all come from the same small group of high schools, where an enlightened administration and devoted teachers fight heroically to maintain high scholastic standards against heavy odds.

We take pride in the fact that some of our high schools send to us a small number of students who have covered essentially our college algebra and trigonometry courses.

More power to these valiant fighters!

In his preface Douglass, who as head of the University of Colorado College of Education and as a writer of secondary school texts is at home in both groups, refers to the *Joint Commission on Mathematics in the Secondary School.*

[2] Possibly the prize answer of all was: "What is the 5th power of 2?"—361,707,296. The student had three years of high school mathematics.

This Commission, and many others, have done outstanding work, and the results of their devoted labor are in print, for all to read. We have also always had on both sides individuals who grasp the problem as a whole, and who counsel wisely and sanely. Why is it that so little has been accomplished by so much effort? Why this ever-widening chasm between secondary schools and colleges? Many of us must feel that the basic reason for the unfruitfulness of so much work lies in the lack of authority enjoyed by any committee we have so far had. Not lack of authority in the sense of lack of confidence in their judgment and wisdom (it is only necessary to allow one's mind to run over the names of the men and women who have carried the burden of such work to eliminate this possibility), but lack of authority in the sense that no organization—neither the large units of the secondary system, such as state or metropolitan units, nor any of the many accrediting agencies—has ever agreed to accept the findings of a commission and promised to put them into effect. We have made a fetish of local autonomy, and it now stands in our path as a well-nigh immovable obstacle.

But even though this difficulty be most serious in character it should be possible to overcome it without encroaching on the established independence of school systems. It seems to me that responsible representatives of the various large high school systems and the colleges of the country and the various mathematical organizations might well be able to agree on the appointment or election of a commission (large enough to be truly representative of all interests, not so large as to be unwieldy), and that a definite program could be set for the commission. On the other hand, the committee should be given assurances that its minimum recommendations will be enforced. For example, it is very unlikely that the commission would be authorized to publish a series of texts of its own; this probably would affect too

seriously the interests of the publishing firms and textbook writers. But it might be entirely feasible and reasonable to authorize the commission to establish a standing committee which will classify all texts according to the purposes they can best serve. Thus, Douglass' book might be rated: "Approved for high school students (Freshmen to Seniors) of limited ability or limited interest in mathematics. Not approved as a substitute for high school algebra or high school geometry," and accredited high schools throughout the country would be under obligation to use it only in this manner.

The crucial point is that we should have a commission which all parties consider utterly competent and trustworthy, and to which all parties are willing to delegate authority to pass binding regulations. The commission might make rules governing entrance requirements for college, and might prescribe a definite minimum level at which college courses in mathematics may begin. These decisions would be binding on the colleges as well as on the high schools.

The tasks facing the commission will be enormous. It will be quite unreasonable to expect this problem to be worked out by busy men and women in their free hours, meeting once or twice a year for a day or two. The members should be freed from all regular duties and be in position to travel and to meet in larger or smaller groups as much and as frequently as is desirable. It might be possible to interest institutions like the Carnegie Foundation in such a plan.

Only if all groups concerned, after they will have had their full share in shaping the commission and formulating a program for it, pledge themselves to accept without reservation the decisions arrived at, can we cope with the hydra whose hundred heads are at present writhing in a hundred directions.

Our school system, instead of following a straight line of progress, has somewhere gone off the track! Anybody's estimate of the underlying reasons is as good as my own, and it is unnecessary for me to express my opinions in the matter. However, it is certain that, once an authentic impartial history of the educational system of the United States will have been written, the whole world will stand before an insoluble puzzle, of understanding how the nation which takes more pride than any other in its school system, which, with immeasurably greater resources than any other country, has poured money like water into its efforts to secure superior education for all its people, has so little to show for so much heroic effort.

Sooner or later, there will be an awakening of the people, and it will then be well for all of us who at various levels have in any measure borne responsibility for the education of the nation to be able to render an accounting of our stewardship.

AMERICAN EDUCATION WEEK 1945

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Education to Promote the General Welfare

Daily Topics

Sunday,	November 11	Emphasizing Spiritual Values
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Mastery of Mathematics Adapted to Needs and Abilities

A Reply to the Previous Article by Professor Kempner

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While naturally I am pleased with the good things Professor Kempner says about me and our book, and while I certainly have no quarrel with the desire to have students come into college classes in mathematics better prepared than they do, I believe that some parts of Professor Kempner's assumptions and reasoning should be challenged and I should like to object strenuously to any criticism of the Douglass-Kinney book, *Senior Mathematics*, for failing to serve a purpose for which it was definitely not intended.

Professor Kempner's principal theses seem to be as follows:

1. Secondary school students should be taught the underlying mathematical principles and theory involved in the mathematical procedures taught, e.g.,
   a. geometrical progressions should not be so taught as to fail to give the fundamental mathematical concept of a repeating decimal.
   b. how to use the general formula for the solution of quadratic equations should not be taught without teaching its derivation and the possibility that it may have unreal roots.
2. Secondary school students in all mathematics classes should be trained in proving theorems used and in the nature of proof.
3. All mathematics courses in high school should be mathematically logical, sequential and systematized and the sequence should be independent of fields of application.
4. Students are coming to college classes in mathematics in alarmingly large numbers without the ability to do a high percentage of types of relatively simple computational and manipulative problems of the nature of such as: Add: $\frac{1}{x} - \frac{1}{y}$; Multiply: 74.5 by 0; add $8\frac{3}{8}$ and $3\frac{1}{2}$; and change price per yard to price per foot.

Professor Kempner objects (and so do I) to the substitution in high school of courses based on a book like Douglass and Kinney's *Senior Mathematics* for formal courses in algebra, geometry and trigonometry emphasizing derivations, theory and proof. I take it for granted that he chose to use our book as an example of a tendency to which he is opposed, not because our book is better or worse than others of its type, but because a copy of our book was at his elbow—a copy presented with my compliments and respects. His constructive proposals are that:

1. The few mathematics teachers who have fought heroically to maintain high scholastic standards against heavy odds should continue to fight on. "More power to these valiant fighters."
2. A national commission be appointed to develop a definite program and that the "commission be given assurance that its minimum recommendations be enforced." Such a commission would "establish a committee which will classify all textbooks according to the purposes which they can best serve."

To join what issues there are between Professor Kempner's thinking and mine and to dismiss without discussion our points of agreement, I shall state the principal pertinent elements in my position.

1. With reference to the commission, most desirable as I believe it to be, the legal and other practical difficulties of giving it power to enforce its rulings are so

great, that I wish to confine my discussion to other points in his article.

2. The high school mathematics teachers are doing a better job than Professor Kempner realizes when one considers the difficulties under which they work today. Their product is less inferior, if indeed significantly inferior at all, than might be expected when one realizes the number of young people entering high school in 1940 was 7,000,000 compared to less than 2,000,000 in 1920 and the center of gravity of their ability to study abstractions has dropped materially. Likewise the number of young people in college in that twenty year period increased from approximately 600,000 to 1,500,000 and there is much reason to believe that the center of gravity dropped at that level also.

3. Teachers in high school are finding it impossible to teach algebra well to ninth grade classes of as great heterogeneity as are at present enrolled for algebra at that level for various reasons—college entrance requirements, influence of parents, advice of teachers, etc., which reasons may or may not be in harmony with the interest or ability of the boy or girl to learn algebra. In the great majority of cases, no more algebra is studied after the ninth grade or any mathematics after the tenth grade and what little is half-mastered is forgotten before college entrance.

4. The decline in the quality of entering college freshman mathematics may be somewhat illusory and difficult to substantiate objectively. As far back as 25 years ago at least there were frequent and very earnest complaints from college professors of mathematics.

It is most likely for reasons like these that the product of the high school algebra and geometry class does not come up to the desired standards of many college professors of mathematics—not because of any degeneration of standards on the part of the high school teachers.

Professor Kempner says, "The argument that our figures indicate that algebra and geometry are above high school intellectual level is refuted by the fact that most of our well prepared students all come from the same small group of high schools, where an enlightened administration and devoted teachers fight heroically to maintain high scholastic standards against heavy odds."

In making this statement Professor Kempner has wandered far afield from the areas in which he is well oriented. As almost any careful student of secondary education could tell him, good mathematics students and poor mathematics students come from all sorts of high schools. Their mastery of mathematics is much less a result of the type and quality of school, than it is of the natural potentiality of the student and the number of years of study of mathematics. To be sure some schools have many more able students and others many fewer than the average school.

It may not be critically pertinent to the major issues involved, but I must make it clear that *neither Dr. Kinney nor I have ever intended that courses based on our book or any other similar course in senior general mathematics be thought of as a replacement for formal courses in high school algebra and geometry.* We do say in our preface, "A student who *masters* a course based on this book will have, as preparation for college study, the *equivalent* of two years of high school mathematics." Professor Kempner refers with misgivings to "the claims of the authors of *Senior Mathematics* that their book satisfies the entrance requirements of institutions demanding a year of high school algebra and a year of high school geometry." *The authors make no such claims in the preface or elsewhere.*

Though *not* mentioned by Professor Kempner—the following important areas are emphasized in our *Senior Mathematics*, decimals, percentage, problem solving procedures, logarithms, the slide rule, solution of the simple equation, solving formulas of various types, proportion, negative and signed numbers, geometric construction with lines, angles, circles, and scaled draw-

ings, precise measurements, approximate answers and data, triangles and many applications of their properties including elementary trigonometry. Throughout we have provided abundant drill, diagnostic and remedial material in arithmetic, more than 100 pages in all on the fundamentals.[1]

I submit, to any impartial mind, that a mastery of these topics with a revived and extended mastery of fundamental computational skills constitutes at least as good preparation for college, other than for the College of Engineering, than is possessed by the *average* entering student with no more than one year each of algebra and geometry taken in the ninth and tenth grades respectively including those of Professor Kempner's beginning students with whom he is so disappointed.

In our book Dr. Kinney and I studiously avoided the teaching of proofs, of derivations, and of any difficult mathematical theory. We concentrated on procedures and skills in fundamentals and procedures of applications. We realized that few, if any, students who had done even acceptably well with algebra would be in classes in which our book would be used. The book and courses employing it are not to be thought of for a moment as substitutes for secondary year algebra and trigonometry. Douglass-Kinney's *Senior Mathematics* is for the student who has not *mastered* algebra: (1) those who have not taken it; and (2) those who were in the lower third or half of their algebra class.

A number of topics were premeditatedly omitted. Not everything could be included in a one year course. If proofs and derivatives had been included the scope of the book would of necessity have been narrower. Those things which were omitted were thought to be either of relatively less practical value in everyday life or to be relatively too difficult to be learned by students in such courses, usually both. Among those the omission of which Professor Kempner objects are factoring of quadratics and a proof of the Pythagorean formula; Douglass and Kinney are willing to leave that issue to the judgment of teachers with recent high school experience.

It is to me almost inconceivable that Professor Kempner or any one else believes that the theory of mathematics, derivations and proofs can be mastered by the great majority or nearly all in high school today. Certainly no one who has taught mathematics in the typical high school today would want to undertake it. That kind of mathematics is difficult, and should not be pottered around with by students in the lower half of intelligence. Teachers of such classes should not be handicapped, as they are today, by having in their classes so many "sow's ears" whose ambitious parents would like to have converted into mathematical "silk purses," that the teacher's time and best efforts are diverted from those who could learn, and as a consequence, the general level of the work of the class adversely affected.

Several things seem quite clear to most of us who are careful students both of secondary education and pupils in general and of mathematics as a school subject, to wit:

1. There should be courses in formal college articulated mathematics in all but the very smallest secondary schools.

2. These classes should be taught by able, well trained mathematics teachers and should not contain many students of less than average capacity for learning mathematics.

3. There should also be other classes in mathematics for those not needing college preparatory mathematics—classes in which the emphasis is upon the more common applications of arithmetic, algebra, geometry and trigonometry to home, shop, farm, health, civics, etc. The everyday world is becoming more and more a mathematical world and the secondary schools must prepare for it.

[1] See "The Place of Mathematics in Secondary Education," The Final Report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics, 15th Yearbook, 1940, p. 25, and Chapters V and VI.

There are some things about Professor Kempner's article that disappoint me.

1. He concentrated his criticisms largely upon Chapter 8 in our book, *Senior Mathematics*—43 pages only—overlooking the fact that that chapter intended largely as a final review of fractions and percentage and a final brushing up of computations, skills before the pupils leave high school for college or for work.

2. He has also selected for criticism as "the basic tendencies and claims of Douglass' new book" such things as our brief treatment of geometrical progressions[2] and the general formula for quadratics, when it is obvious that these brief sections and others like them which are designated by a star are intended only as supplementary material for the bright. To treat such matters thoroughly in the first course in high school mathematics would be obviously impractical.

3. He insists that no mathematical process be utilized in a problem situation without "developing understanding of the mathematical principles and theorems involved." No one text on any level could accomplish this, save for a narrow range of applications. Amost any one application can lead to a volume of abstract mathematics. It is true, as Professor Kempner states, that compound interest may lead to an understanding of geometric progressions, and that the construction of the table on compound interest would lead to a theoretical study of the mathematics of investment. There are many other possibilities that he does not mention—the use of triangles can lead into the abstract relationships of analytical trigonometry; the ratio π into calculus, the average into the mathematics of statistics. It is a commonly accepted fact, among the more careful students of secondary education, that the abstract principles of mathematics should, as far as can be, which would be very far for the type of student for whom our book is written, be developed in this manner. To insist that *all* mathematical understanding be developed from *each* situation, regardless of such considerations as time, grade placement, or the purpose of the course is not realistic. The real problem for the teacher and textbook writer is *which* abstract principles from each situation, *at what level* and *for what purposes*. I do not believe Professor Kempner has thought carefully enough about the problem to appreciate the implications of his position.

With respect to some of the differences between Professor Kempner's thinking and mine, I feel sure that he would agree with me if he could but have a better understanding of what the average and below average high school student of today can be taught. He apparently believes that all should be taught the fundamental theory and reasons why for mathematical computations if the computational procedures are taught at all. *I am as clearly opposed to that position as one can possibly be. It is desirable that the great mass of the population of today and of tomorrow need to be better trained in arithmetical and algebraic computations and their applications to life situations and in the steps and in techniques of problem solving, and I am rather positive that these needs call for training in many techniques the mathematical theory of which is too difficult and time consuming for them to learn.* For example, should the pupil use the ratio π before he has calculus and be able to "change $\frac{1}{3}$ to a decimal" without understanding infinite geometric progressions? It is one thing to have learning of the relationships of the number system, and another to demand "understanding of the underlying principles and theorems involved." We are attempting to teach much mathematical theory now in our conventional mathematics classes in high school and we are failing with the majority of our students. *We are teaching*

[2] In the first printing of our books, a typographical error, in our discussion of geometric progression, slipped by us as will be readily recognized by most teachers of mathematics, i.e., $l = anr^{n-1}$ should of course have appeared $l = ar^{n-1}$. I personally was responsible for failing to catch this error in the proof.

too much and the students are learning too little.

It seems to me that little will be gained by scolding high school teachers and exhorting them to make themselves and their students unhappy by attempting the impossible. Particularly if by so doing they neglect other very important responsibilities.

It seems rather that more serviceable than a Quixotic approach is to survey the situation carefully, realistically, and to develop a constructive program which is in harmony with (1) the needs of the society and time in which we live and (2) the limitations of natural capacity of our students and our capacity for motivating them. It has been that thought that has caused me to contribute in recent years a number of articles on the courses of studies of high school mathematics which have appeared in THE MATHEMATICS TEACHER and other journals.[3]

The substance of these articles involve a double track plan as follows:

1. For the more capable pupils—a logical, sequential series of courses in mathematics either (a) beginning with algebra and running through quadratics in the tenth year, plane and solid geometry in the eleventh year, second year algebra with three months of plane trigonometry in the twelfth year, based upon a required three year sequence of general mathematics in the junior high school, or as next best (b) the old four year sequence based upon a crowded two year sequence in grades 7 and 8.
2. For all other pupils, either: (a) a semester of general mathematics each year—largely arithmetic, algebra, construction geometry without proofs and a little trigonometry, (b) or next best a year of general mathematics of the above type on either grade 9 or 10 and another year in either grade 11 or 12.

It is for the work of the eleventh and/or twelfth grades in this plan that Douglass-Kinney *Senior Mathematics* is intended.

Where the double track plan is intelligently employed, the following results may be confidently expected:

1. All but a very few high school graduates will have had at least two years of mathematics as compared to less than 60% now.
2. The teachers of track one for the older pupils will find their teaching much less discouraging and will be able to devote themselves more to the really capable students.
3. The students following track one, for the abler students will come through with a better mastery of what mathematics they have studied.
4. Those following track two will be able to master fairly well the mathematics most commonly used in everyday life of the average adult and will not quit the study of mathematics discouraged and feeling that it is a waste of time and that it should be abolished from the curriculum, as many citizens do today. (Our boards of education probably include more individuals who either avoided the study of mathematics or quit the

[3] "Let's Face the Facts." THE MATHEMATICS TEACHER, 30: 56–62, February, 1937.

"Two Important Deliberative Reports Concerned with Mathematics in the Schools." THE MATHEMATICS TEACHER, 33: 361–366, December, 1940.

"Mathematics for All." THE MATHEMATICS TEACHER, 35: 212–216, May, 1942. Also digested in *The Education Digest*, 8: 49, September, 1942.

"The Double Track Plan of High School Mathematics." THE MATHEMATICS TEACHER, Vol. XXXVI, Number 2, February, 1943, pp. 17–18.

"Current Trends in the Secondary-School Mathematics Curriculum." *The Bulletin of the National Association of Secondary School Principals*, Vol. 27, Number 112, February, 1943.

"Adapting Instruction in Science and Mathematics to Post-War Conditions and Needs." *School Science and Mathematics*, Vol. XLV, No. 1, January, 1945.

"The Place of Mathematics in Secondary Education." The Final Report of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics, 15th Yearbook, 1940, pp. 25, and Chapters V and VI.

study of mathematics with a bad taste in their mouth or a feeling of its uselessness as taught for the average person than those who take pride in their achievement.)

5. Students who come into college will at least be better skilled in fundamental arithmetic, computations and the simpler algebraic computations and those going into mathematics classes in college will be better trained in mathematical theory and reasoning than they are today.

The war has directed attention to the fact that young people are getting through school today with inadequate mastery of mathematics. According to Navy officers the most serious defects are in the field of arithmetic and first year algebra. Should we not capitalize on the situation and develop courses to meet the demand or is it better to try to sell our critics and our potential customers goods they do not want and can not use?

Shall we see to it that more young people master the fundamentals of arithmetic, algebra, constructive geometry and trigonometry, or shall we persist along the old paths which led us to the present unsatisfactory situation—exhorting to be brave, hold fast on the faith and yield nothing—shut our eyes and ears and plunge ahead in the consoling illusion that we are serving the cause of mathematics in the schools, preserving standards, "fighting off our enemies," or "preserving ourselves from our friends"—even if it does mean fewer and fewer students in our classes and more and more superficiality in the degree of their mastery of mathematics.

Would it be the better judgment to stand on our dignity and refuse to be polluted with arithmetic, life applications, and interested students? Or would it be better to operate on the basis of recognizing that:

1. All high school graduates and former students need to know much more practical mathematics than they now are taught?

2. There are great individual variations among high school students in their respective capacities to learn mathematics and also in the nature of their future needs?

3. Good teaching of the more difficult and theoretical mathematics even to the more capable youngsters cannot be done in classes made up of a considerable number of weak and uninterested students who monopolize the teacher's time and worries and drag down the level of instruction.

I am impatient but not worried. I study carefully the trends in American life and the trends in thought about secondary education. I think I know the thinking of the high school teachers of mathematics. I am quite certain what the ultimate verdict is going to be. I haven't the slightest doubt. I want to live to see it in action. I want to see all students leaving high school with more mastery of the mathematics they will need in life and in college. More than that, I want to have had a part in bringing it about.

Should not the day soon come when college teachers of mathematics, college professors of education particularly interested in mathematics, and high school teachers insist upon understanding each other better—quit carrying chips on their shoulders, cease imputing reprehensible motives or lack of standards to each other, and assuming that his particular group has a monopoly upon a desire to make instruction in high school mathematics more effective. Neither the teaching of mathematics, nor the respect with which these groups are otherwise entitled can but suffer from systematic belligerency and appeal to emotions, superficial thinking, or "class" clannishness. Minds trained to reason should offer more evidence of the effectiveness of that type of training.

Can We Teach Numbers for Meaning?[1]

By George H. Whiteaker

Methods of instruction growing directly from interpretations of the psychology of learning have produced such limited success in the teaching of number that any further proposal to discuss theoretical backgrounds of learning imposes considerable strain on the patience of the practical teacher. Yet such topics must be investigated. A better program for the presentation of number must be found, and if past experience is a criterion, the work must be done with the cooperation of classroom teachers.

The business of learning what makes numbers tick in the human head must begin at the bottom, and follow through grade by grade as far as numbers are taught. The out-moded and over-ambitious psychology of the past twenty-five years began in the middle with the memorizing of number facts, and from all present evidence is dead just there. Upon this foundation of unrelated facts, the structure of functional arithmetic has been purely empirical and opportunistic, because science has had nothing better to offer, and because on the job one can do little more than accept prefabricated methods and materials trimmed and supplemented to meet immediate needs. Yet, it is a bad day for education when a classroom teacher decides that the study of mental processes is futile.

In number, as in no other subject, what one believes about the learning process dictates what things shall be put into a book, or entered into a course of study, and in what order they shall be taught. Adherents to the drill theory of learning offer sets of facts arranged in the order of difficulty, and disarranged with respect to meaning. The drill theory is, of course, based upon the classical laws of learning. Some of the things that puzzle the student of classical psychology are:

1. The so-called laws of learning are not concerned with learning at all but with the memorizing of nonsense syllables.
2. Learning is considered as an act of the intelligence, and classical psychology warns the student against mistaking memory for intelligence.
3. Every classical psychology includes a chapter on forgetting and shows the curve of forgetting which points the way to oblivion for everything that is learned by repetition alone.
4. No classical psychology even suggests that meaningful facts should be deprived of their meaning by dissociation. The idea of learning for meaning is not exclusively the property of the Gestaltists. James wrote in 1892, "The one who thinks over his experiences most and weaves them into systematic relations with each other will be the one with the best memory."[2] Thorndike has said,

> The newer pedagogy of arithmetic then scrutinizes every element of knowledge, every connection made in the mind of the learner, so as to choose those which provide the most instructive experiences, those which will grow together into an orderly rational system of thinking about numbers and quantitative facts.[3]

In 1927, Dr. Judd gave this warning,

> If arithmetic is taught as a series of wholly detached special rules of combination, the pupil is deprived of the most valuable aid to memory and scientific thinking that the mind possesses.[4]

Primary concepts of number are a familiar topic in the literature of education, and primary teachers have been doing an ex-

[1] Reprinted in part from: A Growing Concept of Number, Bulletin of the Mathematics Section, Eastern Division Colorado Education Association, Fall 1944.

[2] James, William, *Briefer Course in Psychology*, Henry Holt, New York, 1892, p. 294.

[3] Thorndike, E. L., *The Psychology of Arithmetic*, New York: Macmillan Company, 1922, p. 74.

[4] Judd, C. H., *Psychological Analysis of the Fundamentals of Arithmetic.* Chicago: University of Chicago, 1927.

cellent job of establishing cardinal and ordinal meanings and of establishing each number's place in the primary addition combinations and the multiplication tables. There the matter has rested. The majority of college freshmen bring with them those primary concepts of number and little else, simply because no one has proposed; indeed, no one knows, just what an adult's concept of number should be. Five years ago it was easy to outline a primary concept of number and to document the study from current and near current literature.[5] Yet, today, five years later, proposals for a larger concept must stand practically unsupported.

When there is no other attack on a learning problem, the historical or recapitulation theory may give us an entering wedge, and in the present situation a reference to the way in which the science of number first developed in the mind of man may be in order. The picture is not clear, but we are quite sure that man first recognized groups of two, three, and possibly four, and then began counting on his fingers, first with one hand, and then with both. Hence, he had a "two-times-five system." Even after he learned to represent two hands by one finger, he was still thinking "two-times-five." It was thousands of years before he saw it as a decimal system. Examine the Roman numberal system in this light.

Five fingers	IIIII equals V one hand
Two hands	VV equals X a pair of hands
	XXXXX equals L
	LL equals C
	CCCCC equals D
	DD equals M

So far no one thought of memorizing 45 addition combinations. The Roman student did not spend long hours inscribing the 45 combinations on sea shells. He saved his pennies and bought an abacus. This simply furnished an easy way for trading five ones for a V, two V's for an X, etc. You can do the same thing on paper if you write the Roman numerals vertically, without the subtraction device, and then add up from the bottom. As the centuries rolled on the clumsy Roman symbols were displaced by the Arabic numerals with a different form for each number from 1 to 9. After 9 there was considerable trouble. Such awkward numbers as "5 and twenty," or "four score and seven" have persisted to almost modern times. The idea of place value was invented or borrowed from the Hindus, but there was still no way to write such a number as 2,000 or 206 except in words. Then someone put a ring around nothing, named it ZERO and gave it the duty of holding the empty places. With this invention the two-times-five system became in reality a decimal system. As results of this development we have the 45 combinations and the multiplication tables, but we lost the abacus. The trade was not all gain. The abacus shows at a glance some of the things which a modern college student does not know—the peculiar relations of 2, 5, and 10. The average college student knows that 1,000 is the cube of 10. Remind him of this; then remind him that 10 is the product of 2 and 5, and he is puzzled when told that for these reasons, 1,000 must be divisible by 8 which is the cube of 2; and baffled when told that the quotient of this division must be the cube of 5 or 125. Further, he may perform multiplications of 5 times small even numbers a dozen times in a row without discovering the shortcut, and he is likely to test such a number as 138 before pronouncing it indivisible by 5.

In fact, the whole field of factorial relationships seems to be unexplored ground. Students are usualy surprised to find that the product of 4×28 is divisible by 8. They are unable to say without testing that the product of $17 \times 17 \times 17$ is not divisible by 11 or 13. Many fail to realize that a number ending in 5 cannot have a divisor ending in 4, and they see no reason to expect the product of 4×36 to be the same as that of 12×12. They are puzzled by the

[5] Whiteaker, G. H., "A Child's Concept of Numbers." THE MATHEMATICS TEACHER, January, 1939. XXXII: 25.

statement that $1\frac{1}{4}\times 16$ can be had by adding 4 to 16 and they require considerable coaching to discover that there is a relation between the two facts, $1\frac{1}{2}\times 8=12$ and $15\times 8=120$. Even on the more familiar ground of common denominators there seems to be little understanding of the processes used.

There is no intent here to censure anyone for failure to teach these things—they have never been prescribed. As facts these items are not important, and the pattern to which they belong has never been drawn and analyzed for the purpose of teaching. It is regrettable that textbooks, courses of study, and even lesson plans are organized along lines dictated by the drill theory of learning in spite of frequent professions that we no longer believe that drill offers the real answer to number difficulties. A great deal of careful work was done during those decades when the teaching profession appeared to forget every chapter in the psychology book except the one on rote learning. The most voluminous literature was written, the most elaborate teaching materials were prepared, the most scientific-appearing texts were printed, and the most exacting tests were set up and standardized with the drill theory as a background. In view of these conditions it is not surprising that college entrance tests still show that incoming students have been well taught in matters that can be taught by drill, and not so well in matters that require insight, understanding or generalization.

To date, teaching of mathematics for understanding has been generally practiced only at the primary level, where it has been abundantly justified by results, and in departments of higher mathematics where an attempt to learn by repetition becomes absurd. Upgrading of material, use of practical problems, functional units, socialized projects, and allowances for individual differences are all steps in the right direction. No doubt each of these methods contributes to understanding, if only by delaying the work until understanding has a chance to develop, and that is all that can be said, in fact, that is all that is claimed for some of the innovations of the past decade. We are still a long way from the position taken by Dr. Judd in 1927.[6] The establishment of number concepts, however elaborate they may be, and the use of numbers in problem situations, however real they may be, does not satisfy the dictum that the number system is to be presented as a coherent, orderly scheme of thinking.

The job of reorganizing the number facts into a meaningful system for the purpose of encouraging generalizations and transfer was begun in Detroit some fifteen years ago.[7] The experiment included only the learning of the primary addition facts, but strong evidence was produced to indicate that these facts are more effectively learned when presented in a rational order.

To reorganize the entire field with the same purpose in mind will be a tremendous job, but until it is done no real test can be made of the proposition to teach number for meaning. A broad view of the meaning theory requires that every item and every topic in each succeeding grade be presented with regard to its place in the system. Some facts and processes which have been trimmed away by previous revisions may demand reconsideration, and some items which have been emphasized may be found relatively nonessential.

By way of illustration and in the hope of starting discussions we offer a few random observations. We have no intention of defending these propositions, but think that each is worth a trial in its proper setting in a numbers program that is based upon meaning. We shall expect no results from insertion of one or more of these ideas into a course that is based mainly upon drill.

1. There is sufficient evidence to justify grouping or classifying the number facts for presentation.

[6] Judd, C. H., op. cit.

[7] Thiele, C. L. *The Mathematical Viewpoint.* Tenth Yearbook, National Council of Mathematics Teachers.

2. The addition facts should be followed by multiplication rather than by subtraction. Teach additive subtraction if it seems most effective, but present it as a new mental process and far enough away from addition to avoid interference.

3. Zero difficulties are aggravated by considering zero as a number. Its primitive concept of "placeholder" is adequate for elementary work.

4. Sooner or later every number below 100 should be learned as a prime number or as a product of primes. Sooner or later the multiplication facts should be presented in tables.

5. More attention should be given to factors, multiples and reciprocals, with especial attention to 5, 25, and 50. This implies a more careful study of the meanings of multiplication and division and may indicate the return of cancellation as a technique in problem solving.

6. A meaning for multiplication that is based upon counting things in rows and layers is a convenient rationalization for beginners, but is inadequate for the upper grades. With problems in mensuration should come the ideas of dimension and degree. The student who has to distinguish among formulae for length, area, and volume by memory alone has but little understanding of multiplication.

These few proposals may be suggestive of the amount of work that lies ahead. After the entire field has been reorganized to provide real living concepts, meaningful operations and significant situations in orderly sequence, then we shall be ready to try the proposition of teaching number for meaning.

Mathematics*

(With Apologies to Shakespeare)

Friends, neighbors, classmates, lend me your ears.
I come to talk of Mathematics; not to praise it.
The poor marks that we get go home with us,
The good are all forgotten by our teacher.
Thus it is so, with me, the speaker.
Our good instructor tells us to be ambitious.
For if we are, we'll get the extra credit,
And grievously have we pupils answered her.
Here, now, before our teacher and us pupils,—
For our teacher is an energetic woman.
So are we all, all energetic people—
Come I to say, Don't study mathematics!
It is an awful subject, nothing else but figures,
Yet, still is praised and championed by our teacher;
And sure she is a truthful woman.
She had taught many people how to add,
And think in terms of squares and angles;
which makes our brains both quick and nimble.
When marks are low, we sadly all have wept.
Oh! dear! Arithmetic should be abolished!
Judge, O Ye Gods! how hard we strive to learn it.

* The above poem was written by Ina MacFarquhar, a pupil of 8B grade in Alexander Hamilton School of Elizabeth, New Jersey.

Changing Philosophies in the Teaching of Mathematics

By Olive Leskow
Tolleston School, Gary, Indiana

"Mathematics, the schoolboy's horror, is perking up again after a long sabbatical in the educational doghouse."[1] This statement surveys the psychological trends of the teaching of mathematics in the past two centuries. Why was it the "schoolboy's horror"? Why was it in the "educational doghouse"? Why is mathematics again "perking up"? And, will it continue to "perk up" after the war? To answer these questions it is necessary to give thought to the errors of our predecessors so that we will not repeat the same errors and again shove mathematics into the doghouse.

In most educational studies we usually start with the question of objectives. Being self-centered individuals we state our objectives in terms of our subject matter, feeling that without our subject-matter field the child would be seriously handicapped. However, in this paper I choose to attempt to think in terms of general educational objectives and how they can be obtained through the study of mathematics in its broader sense.

In 1819, a textbook entitled, "The Scholar's Arithmetic," was printed in New Hampshire.[2] This book, although printed in the nineteenth century, was of a work book nature. It featured meaning and understanding. A few quotations from this workbook deserve to be mentioned. The author stated:

"Remember that youth, like the morning, will soon be past, and the opportunities once neglected, can never be regained."

"As much as possible, endeavor to do everything of yourself; *one* thing found out by your own thought and reflection, will be of more real use, than *twenty* things told you by an instructor."

"Understand everything as you go along."[3]

We shall come back to these quotations and see how Daniel Adams accepted the so-called modern view of general education. It is interesting to note that as early as 1819 true educators had the general educational objective in mind rather than subject matter idealizations.

In 1827 evidence shows that in Massachusetts, algebra, geometry and surveying were required in every high school in towns of 500 families or more.[4] What were the reasons for such a requirement? It was presumed to have a disciplinary effect on the adolescent mind.

When this result was questioned, educators cited the traditional standing of mathematics as an element in cultural education and its usefulness in the practical affairs of life.[5] Thus, three objectives have been emphasized, namely, disciplinary, cultural, and utilitarian. Do these contribute to the general education? Yes, if interpreted in that manner; however, again our selfishness seems to play a part in interpreting these with our narrow outlook.

In the early nineteenth century the disciplinary concept was predominant in all countries. In Tolstoy's "War and Peace" we find Prince Bolkonski speaking to his daughter: "Mathematics are most important, madam! I don't want to have you

[1] "Third R," *Time,* XXXIX, 39.

[2] *The Scholar's Arithmetic* by Daniel Adams was printed by John Prentiss of Keene, New Hampshire in 1819.

[3] Ben A. Sueltz, "Adams Did It 125 Years Ago: Salient Features of *The Scholar's Arithmetic,*" The Mathematics Teacher, XXXVI, 183.

[4] Progressive Education Association, *Mathematics in General Education* (New York, 1940), 3.

[5] *Ibid.*, 4.

like our silly ladies. Get used to it and you'll like it. . . . It will drive all the nonsense out of your head."[6]

In 1893 the first comprehensive survey of the program and purpose of secondary education in the United States assumed that mathematics had a general disciplinary value. Today there are still certain educators who fight constantly to keep the disciplinary value in the minds of teachers. Brueckner states that the continuance of arithmetic in the curriculum is increasingly justified on the grounds of mental discipline rather than its practical utility, since the arithmetic course has developed into "an unwieldy mass, containing a body of content much of which is useless and impractical."[7]

At the turn of the century the utilitarian concept came to the foreground. Educators realized that many of the techniques taught in their textbooks were no longer needed because of the changes in the economic and social customs. Recommendations were made that arithmetic learning exercises and teaching methods should be made to agree more nearly with that which was being used, or at least, could be used profitably. Arithmetic was thus considered a tool subject.

The cultural concept is getting more attention today. Although recognized, it has never been accepted as the major objective, since there is considerable doubt in many minds as to its contribution to education.

The changing of objectives has also been affected by educational trends. From the beginning of the twentieth century the elementary school curriculum has been in the process of being broadened in scope. Consequently the time element for formal arithmetic has been decreased.

The testing movement in the teens of the century has put much stress on the computational aspect of arithmetic. Computation was emphasized to almost a complete neglect of other important values. As a result there came a decline in the students' abilities to apply numbers to life situations. "Their number knowledge," as one writer puts it, "seemed to be in the air and of little use."[8]

Curriculum directors began to feel that formal numbers were undertaken too soon and recommended that arithmetic in grades one and two should be incidental. However, incidental learning is effective only under the guidance of skillful teachers. Since many teachers almost completely neglected the teaching of quantitative feeling, there was insufficient learning.

Incidental learning, however, made two valuable contributions. It emphasized children's interests as a motivating principle for effort. It also stressed the value of concrete illustrative material for giving meaning to learning.

During the past ten years, the social aspect of general education has been greatly emphasized. In the field of mathematics it is even exaggerated. At the outbreak of the war, army and navy officials found men so weak in mathematical skills, they couldn't qualify for officer training. This condition implies too much stress having been placed on the social aspect of teaching. With this condition in mind, perhaps we should develop a course in mathematics that will not overemphasize the army and navy demands to the neglect of peacetime objectives.

We must recognize that equal weight should be given to both the social and utilitarian aims. We must not forget that understanding, which was preached as far back as 1819, is the underlying aim of mathematics. We must also remember that shifting the work to a later maturity stage of the child's life may deprive him of something for which he may already have a need. New evidence seems to point out

[6] Leo Tolstoy, *War and Peace* (New York, 1942), 94.

[7] Leo J. Brueckner, "The Social Phase of Arithmetic Instruction," *Arithmetic in General Education* (New York, 1941), 141–142.

[8] E. A. Bond, "A Proper Balance Between Social Arithmetic and a Science of Arithmetic," The Mathematics Teacher, XXXV, 314.

that the child is maturing more rapidly—rote learning can be prevented if we have well-qualified teachers. At all stages of working with children we should question ourselves as to whether or not we are practicing teaching methods that will develop their personalities as a whole. Will they be better people tomorrow because of what they do today?

With these objectives in mind, we can incorporate good teaching methods from the old trends in a course which will be both modern and educational. In this connection one contribution of the former trends is that of remembering the child's interests. Our course should start with the interests of the child. The child's problems should be studied. All children meet certain problems when they strive to meet their need in the basic aspects of living. Such simple problems as selecting their own Easter outfits are more interesting to them than those presented in the textbook. The mathematics course should, therefore, help the pupils solve such problems with mathematical concepts and methods. The pupils' attention will be held much more easily since the problems are in the field of their interests. But do all of the teachers of today follow this plan? It seems that in most classrooms the old plan is favored—that of the drill theory.

Where did this drill theory originate? It probably arose out of the great emphasis on computational tests in the teens of this century, where it was felt that the repetition of the same types of problems would help the individual remember the processes and facts. In spite of this practice, progress was not shown. Too many teachers of that period found that using the drill theory made arithmetic teaching easy. It was easy merely to assign daily practice and keep the pupils busy.

Our second suggestion in building up our modern course is that of making arithmetic meaningful. The modern educational theories stress meaning, because understanding of the thing learned both increases the opportunity for recall and at the same time strengthens the significance of that which is being learned. Both of these results help children acquire attitudes favorable to learning. Bond, in considering the importance of meaning, states: "One cannot use the skills he possesses unless he understands their significance in attacking situations that contain quantitative aspects."[9]

Teaching for meaning should give rise to many methods or a method combined with a variety of methods. To give meaning to arithmetic we must teach for various aspects of meaning. Take for example the teaching for quantitative thinking. Quantitative thinking is number value thinking. In the report of Mathematics Committee of the Progressive Education Association it is suggested that quantitative thinking should be obtained through data-collecting projects.[10] The teacher should be on the alert to note the various occasions which will warrant data collecting. Surveys of opinions, only, do not contribute as much as do the more elaborate projects of data collecting nature. Of course, the type of problem selected should be named by the pupils.

Last year, my class brought up the question of whether or not using one's own car to ride to and from work is less expensive than using the street car or bus. Various views were expressed. Here was a problem worth gathering data on. At first, students should be shown that facts should be gathered indiscriminately. The next step is to inspect the facts and see what conclusion they may point to. Then will come the time to formulate the problem. In the inspection the student will acquire not only quantitative thinking but also reflective thinking. The problem having been formulated, the student is then ready to form his conclusions.

The greatest handicap in using this type of teaching is our present curriculum. Each

[9] Bond, *op. cit.*, 315.

[10] Progressive Education Association, *Mathematics in General Education* (New York, 1940), 90–109.

teacher is faced with a certain amount of subject matter which she must cover. The ingenious teacher will not let this disturb her, but will look into another important aspect of thinking. Some data collecting and such problem solving can probably be taken up if other units are given less time. Indirectly, studying relationships contributes to an easier understanding of the field of numbers and, thereby, saves time. Many psychologists feel that teachers do not point out relationships often enough. Professor C. J. Keyser states his view on this topic: "Everything in the world has named and un-named relations to everything else. Relations are infinite in number and kind. *To Be Is To Be Related*."[11]

He goes on further to show how important to the lives of individuals is the problem of relationship: "It is evident that the understanding of relations is the major concern of all men and women."

Then, are relations the concern of mathematics? They are so much the concern that mathematics is sometimes defined as the "science of relations."[12] Wheat, in his definition of arithmetic considers it a system to be acquired by a study of number relationships.[13]

Christofferson has done much work stressing interrelationship in junior high school mathematics. Besides stressing interrelationships he also emphasized the fact that pupils should be made to appreciate the abstract values of mathematics before the topic is completed. Let us look into some of the suggestions which he makes about teaching various topics.

How can the area of the circle be taught so as to show interrelationship? It is taken for granted that the pupils have had the circumference formula and understand it. Let us take a circle and cut it into sectors. If the circle is divided into millions of sectors and these are spread so that the circle is bent out into a straight line (it will appear as such), the result will be a series of triangles with the radius as the common altitude and the invisibly curved arcs as bases. If we take half of these triangles and fit them into the other half, we will have a figure which will approach a parallelogram. We can observe that the base of this parallelogram is equal to $\frac{1}{2}$ the circumference. Since the area of a parallelogram is the product of the base and the altitude, the area of this parallelogram is the product of $\frac{1}{2}C$ and r. But C is equal to $2\pi r$. By substitution, the area of this parallelogram is equal to $\frac{1}{2}(2\pi r)r$, or πr^2. Here we have arrived at the formula of the area of a circle through interrelationships.

Now let us see what relationship there is between the area and the radius of the circle. What happens if the radius changes? Would only the height of the parallelogram change or would the area change also? If we increase the radius we can establish a relationship that the area of a circle increases. Therefore the area of the circle is dependent upon the radius.

Christofferson offers many other suggestions in both the pure mathematics field and the business mathematics field. His suggestions can give us ideas of how to approach any topic and point out relationship and also the abstract values of the topic. Here definitely, we can show how one value depends upon another. If we wish to go farther we can with this approach teach one parameter and two parameter data, etc.[14]

This may lead into what may be classified as teaching for functional relationships. However, most teachers disagree on the meaning of the word, "function." Cronbach found this to be true in his study, but agrees that many people who know the meaning of a certain term may not be able to explain the definition but may be able to use and apply it properly.[15]

[11] C. J. Keyser, Male Philosophy and Other Essays (New York, 1927), 94 ff.

[12] *Ibid.*, 95.

[13] The National Council of Teachers of Mathematics, *Arithmetic in General Education* (New York, 1941), 80–118.

[14] Halbert C. Christofferson, "Teaching Relationship in Junior High School," THE MATHEMATICS TEACHER, XXXIV, 343 ff.

[15] Lee J. Cronbach, "What the Word 'Function' Means to Algebra Teachers," THE MATHEMATICS TEACHER, XXXVI, 212 ff.

Another rather important suggestion for our new course is that of emphasizing generalization. Buswell, at a meeting in Chicago, listed the weaknesses of present day arithmetic.[16] One of the most outstanding is the failure to appreciate the value of an *abstract* use of numbers in meeting the needs of life.

The development of this was shown in the illustration from Christofferson. The acceptable theory is that abstract practices with numbers should follow the concrete experiences. The abstract relationships grow out of the understanding of the concrete. But again, in most modern classrooms there is a tendency to stop with the concrete.

"An abstraction is a generalization that grows out of concrete experiences."[17] The child should be shown that the value of abstraction is found in the fact that it short-circuits the awkward process of using concretes in our thinking. It is just as important that arithmetic should be carried through to the point of abstraction as that it should begin with the point of concreteness. Abstract concepts are not arrived at incidentally.

What objective is fulfilled by this type of teaching? If we mean general education, can we say that this type of teaching concerns itself primarily with the highest possible development of the individual's personality? Again, the teacher must keep in mind that in order to benefit the individual, he must think in terms of personality needs and the effects of his demands and practices upon that person.

All experiences affect personality and these in turn are affected by personality. It is important to consider how the individual might see himself in comparison to what we see in the individual. We cannot drift away from mental hygiene in working with individuals. Subject matter and skills are merely means to that end.

At present it seems very easy to influence adolescents to take all the mathematics that is offered in our high schools. The reason back of this lies in the fact that our periodicals have widely publicized the army needs. Boys home on leaves emphasize the importance of the subject in the armed forces. Immediately the younger generation acquires a favorable attitude towards the subject. Likewise, educators have been influenced by this publicity. They, too, are projecting their views and influencing the acceptance of mathematics as a utilitarian subject.

But we must not forget that objectives change with the times. Too much stress upon this element will not prepare the individual for what he might need. It must be remembered that one cannot use knowledge that he does not possess. Sometimes it is necessary to teach elements that do not have immediate practical value but the use of which will prove valuable. If major premise was placed on thinking, and the tools and skills given as means to obtain such thinking, our curriculum probably would not suffer from today's comments. As the attitudes of both educators and students change with Victory, we must keep up the general value of mathematics as a means by which we can make better thinking citizens of our youth.

In our educational field we should add knowledge and facts to our theories. We should organize and develop better tests, which will test not only computational skills, but the abilities to apply number needs and to see number elements in complex situations. Learning then will be filled with meaning. Arithmetic will become a mode of thought enabling the citizen to analyze his number elements in response to whole situations.

Perhaps then the youth may solve our peace problem and make peace everlasting.

[16] Guy T. Buswell, "Weakness in Present Day Arithmetic Programs," *School Science and Mathematics*, XLIII, 201 ff.

[17] *Ibid.*, 202.

Better Teaching of Mathematics

By Elizabeth Beaman
Columbia, South Carolina

Although I am not a teacher of the calculus, but of high school algebra and geometry, and have had only limited experience even there, I should like to comment on an article in the December 1944 Mathematics Teacher, "The Teaching Objectives in a First Course in the Calculus," by James E. Parker.

The three major objectives which Mr. Parker sets forth in this article are excellent. Although he states these points and discusses them with especial reference to the teaching of a first course in the calculus, I feel that they are also very applicable to the teaching of any other branch of mathematics. It seems that the three points would be especially significant to teachers of mathematics in our public schools as well as in our colleges.

Mr. Parker's first objective, "to give the student an understanding of the fundamental concepts of the calculus and a point of view relative to the historical background out of which these concepts grew," is definitely an excellent approach to a course in algebra, whether beginning or advanced. Algebra, like the calculus, contains a number of concepts which must be clearly understood before a student can master the subject. This same objective might well apply to geometry and trigonometry in which there is a need not only for understanding certain concepts, but also for visualizing these concepts.

Perhaps this first objective is the solution to the arithmetic problems of the high school teacher. There are many students enrolled in algebra or geometry classes who are amazingly weak in arithmetic, and there is much discussion about whose place it is to teach the fundamentals of arithmetic. If more stress is placed upon understanding the concepts rather than upon memorizing a fixed procedure, this problem which causes further study in algebra to be ineffective might be partially solved.

It could be that we, as public school teachers, have put too much stress upon Mr. Parker's second objective, "to develop proficiency in the manipulative skills of differentiation and integration." Again applying this principle to the less advanced courses of high school, we often find that students know how to perform certain solutions only because they have memorized the procedure. Give them the same problem differently arranged, and they are completely lost. I do not mean to imply that this second objective is not important, but that it is very often overemphasized. It is easy for a teacher to assign procedures to be memorized without requiring any real thought on the part of the student. Therefore, we must be very careful to stress proficiency and to help pupils to acquire skills without memorizing rote facts.

The third objective, "to develop abilities in making practical applications of the principles learned," is very frequently neglected, especially in our high school mathematics courses. Most students shrink from any problem stated in words, merely because they have memorized their algebraic procedures and do not see any connection between these English sentences and the procedures which they have previously learned. Problems should be taught throughout any mathematics course and used as a means of acquiring a better understanding of fundamental principles through practical applications.

For the above reasons, I feel that these three principles should be set before teachers of arithmetic, algebra, and geometry as well as teachers of the calculus.

In general, mathematics is not a very

popular subject in our schools. In times of peace, proficiency in arithmetic is not stressed sufficiently in the elementary grades, students are not encouraged to take algebra in the high school, those who take a year of algebra and then go on to advanced algebra, college algebra, or trigonometry, are few, and college students, as a group, take as little mathematics as possible. It is only in extreme emergencies such as the present world conflict that mathematics really assumes the place that it should hold in our system of education. I wonder how much of the responsibility for this peace-time lag can be placed upon our teachers and upon our methods of teaching.

The majority of pupils enrolled in high school mathematics courses are not there because they have a passionate desire to study the subject. It is, therefore, the responsibility of the teacher to make the subject real to the pupil, to make it understandable and reasonable to him. The important place that mathematics holds in war time aids the teacher along this line, for most students are interested in some aspect of the war. Most of them have brothers, fathers, or friends in the Service, they have been limited by our system of rationing, and they have taken a very active part in the sale of war bonds and stamps. These things have meaning for them, and help them to realize the seriousness of the war. When they are taught the place of mathematics in the war effort, then mathematics becomes something real and vital to them in their lives.

Our problem of motivation, therefore, needs much more consideration in peace time than at the present time. I believe that if the three principles which Mr. Parker advocates for teachers of differential and integral calculus were kept in mind by all teachers of mathematics, and were properly emphasized in all grades and in all mathematics courses, we might find here the solution to one of our peace time problems. In the future, we might succeed in offering our children and young people a well-rounded mathematical education which would be both useful and interesting to them.

Arithmetic Achievement of Japanese-Americans

By Mrs. Harriet C. Pusey
Indiana University
Bloomington, Indiana

There appears to be a general impression that Americans of Japanese ancestry possess unusual aptitude for mathematics. The impression also persists that while students in our schools of this ancestry excel in fundamentals or routine mathematics, they are slow thinkers, therefore, fall behind other American students in reasoning ability.

In the effort to determine the truth or falsity of this belief the following study was made. In December, 1943, 484 junior high school students in the Japanese Relocation Center at Manzanar, California were tested by the *Metropolitan Advanced Arithmetic Test: Form B.* Unlike other situations where these students have been tested in public schools in competition with other racial groups, these students have a common ancestry. As far as the writer is able to ascertain, no accurate data on arithmetic achievement has been compiled for Americans of Japanese ancestry. Manzanar High School is fully accredited by the California Department of Education. All of the students are Americans of Japanese ancestry, therefore, present a school situation unlike any other to be found in the United States outside of the ten Relocation Centers.

Of these there were 125 seventh graders, 117 eighth graders and 242 ninth graders. The ninth graders were members of the General Arithmetic classes and Elementary Algebra classes and included 39 tenth graders, 18 eleventh graders, and 15 twelfth graders—a widely assorted group.

The test was made up of two parts: *First Fundamentals* covering multiplication and division of both common and decimal fractions, percentage and mensuration. Second, *Problem Solving* or Reasoning with written problems.

In order to interpret results of this test with any degree of accuracy it is necessary to take into account the school histories of the Manzanar students over the past two years.

Evacuation from their former homes and removal from their school environment took place in March and April of 1942. At least three months of the regular school session was lost that school year. School organization that would meet requirements for standard school work at Manzanar was delayed and later during the period known as "the riot" was stopped altogether during the fall session of 1942 so that approximately three to four months of school time was again lost. Therefore, it seems a conservative estimate to say that the students taking the arithmetic test should be allowed a differential of at least six months in comparing their scores with standard scores for the same age levels.

Comparing of Median Grade Equivalents for Fundamentals and Reasoning

GRADE	Standard Norm	Fundamentals	Problem Solving	Average	No. of Months above Grade Norm
7	7.3	7.3	6.9	7.1	−2
8	8.3	8.4	7.7	8.0	−3
9	9.3	8.7	8.3	8.5	−8

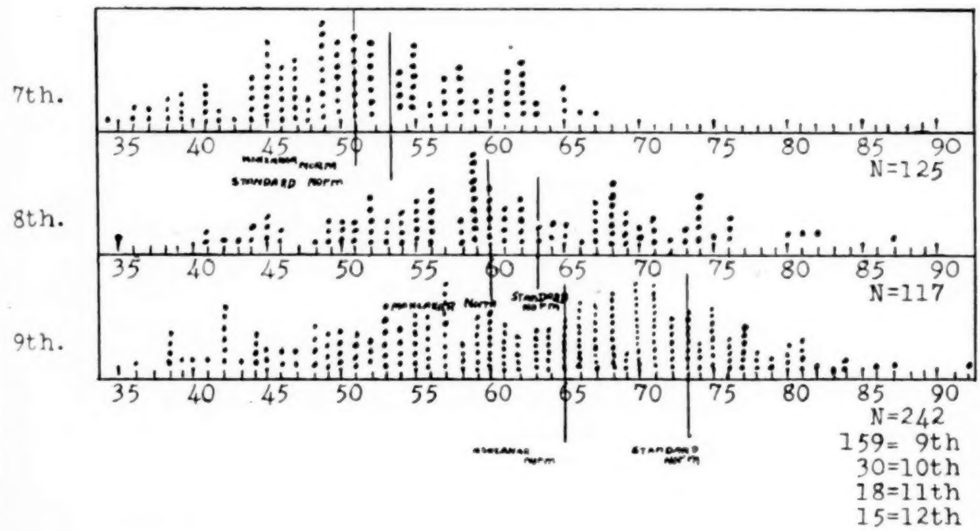
In all three grades tested the median grade placement for Fundamentals was higher than the median grade placement for Problem Solving. The seventh grade

median was 7.3 or exactly with the standard norm. The eighth grade moved up a month beyond the standard norm to 8.4. The ninth grade which included sixty tenth graders, eighteen eleventh graders, and fifteen twelfth graders scored a median of 8.7 or 5 months below the standard ninth grade norm.

In Problem Solving or Reasoning all three grades fell below their grade norms. The seventh grade scored 6.9 or 4 months below; the eighth grade 7.7 or 5 months below, and the ninth grade, including students from the ninth grade to the twelfth grade, 9 months or approximately a full school year. The ability to reason reached its highest point in the eleventh grade where the median for the eighteen students tested with the ninth grade had an age equivalent of 14.7. The fifteen twelfth graders included in the test scored a grade equivalent of 8.5. In Fundamentals the sixty tenth grade pupils attained a grade equivalent of 9.2, the highest point scored by any group tested. The low scores made in Fundamentals by the eleventh and twelfth grades may be partially explained by the fact that two or three years have intervened since these individuals have had any class in mathematics and mathematics on the first year high school level, or the ninth grade, had been elected this year not because of interest in the subject as much as to fulfill requirements for graduation.

The Scattergrams for all three grades show a wide spread. The seventh grade ranges from 5.4 to 8.7, a spread of 3.3 years, the eighth grade from 5.5 to 10.7, a spread of 5.2 years; the ninth grade from 5.6 to 11.2, a spread of 5.5 years.

SCATTERGRAM OF SCORES



The ninth grade shows a greater cluster of scores between the Manzanar norm and the Standard norm than is true of the seventh and eighth grades. There is also a wider spread between these two norms showing a larger number of cases slightly below the standard norm, but above the class norm. Twelfth grade students taking the test rated lower in reasoning than the

SAMPLINGS OF TWELFTH GRADE STUDENTS

	Chronological Age	Grade Placement	Age Placement
11th Grade	20	9.8	15.1
12th Grade	19.1	9.2	14.7
12th Grade	20.8	7.7	13.2
12th Grade	19.4	7.3	12.10
Post Graduate	19.2	8.0	13.6

class norm. Over-ageness was typical of this group.

Summary and Conclusions

Results based on the Metropolitan Test show Japanese American students of Junior high school grades of the Manzanar High School to be slightly below their grade level in the composite score. When broken down into the two test divisions, it was found that the seventh and eighth grades were slightly above the standard norm in Fundamentals but below in Reasoning. The ninth grade made up of students from the ninth to the twelfth grades rated a year below. The twelfth graders did not show increased ability in reasoning over the other grades. When allowance is made for time lost from school the class medians are above the standard medians. From this it would appear that these students do possess aptitude for fundamental processes. Scores obtained on Problem Solving appear to indicate that the ability to apply reasoning to Problem Solving is slightly below other students of the same grade level.

Section One Meeting

ASSOCIATION OF MATHEMATICS TEACHERS OF NEW JERSEY

Saturday, April 28, 1945

Newark Athletic Club, Newark, New Jersey

Section One Officers

President, Roscoe P. Conkling — Secretary, Dr. Amanda Loughren

J. Dwight Daugherty, State President, presiding

10:00–10:05	Greetings—Roscoe P. Conkling Barringer Evening High School, Newark, New Jersey
10:05–10:20	Message from Our State President J. Dwight Daugherty Eastside High School, Paterson, New Jersey
10:20–11:05	Study of the Teaching of Post-War Mathematics in New Jersey Dr. Howard F. Fehr, General Chairman State Teachers College, Montclair, New Jersey
11:05–11:50	Changes in Mathematics Due to the Current Tendencies Dr. Foster E. Grossnickle State Teachers College, Jersey City, New Jersey
11:50–12:10	The National Policies Commission Dr. Virgil S. Mallory, Our Association's Representative State Teachers College, Montclair, New Jersey
	Luncheon—1:00 p.m. Newark Athletic Club
2:00	The National Council and Post-War Mathematics Dr. F. Lynwood Wren, President National Council of Teachers of Mathematics George Peabody College, Nashville, Tennessee

Due to wartime restrictions, those desiring luncheon reservations should write to Mrs. Florence Borgeson, Roosevelt Junior High School, Westfield, New Jersey as soon as possible. Price of Luncheon—$2.00.

The Graphic Solution of a Problem Involving Simple Linear Equations

By W. W. Ingraham

The Williamstown School, Williamstown, West Virginia

The average high school pupil is mature enough to realize that problems about Mary and the postage stamps, and grocer Brown and his mixtures of coffee are downright nonsense. There are very few algebra text-books which are not heavily laden with such problems. In order to get a wholesome response from the class, problems must have a purpose or reasonable objective. The problem offered herewith is only one of many, when solved by the pupil, tend to increase his respect for mathematics rather than lower it. The Problem: The factory, home workshop, and Industrial Arts Shop are the source of

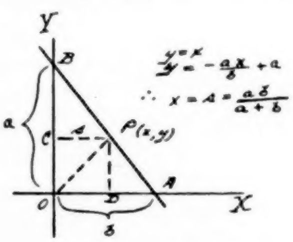


Fig. 1

scrap material, such as: ends from sheets of metal, plywood, etc. Some of these pieces are in the form of right triangles, others are in the form of trapezoids. The class is asked to determine a method of obtaining the dimensions of the largest square that can be salvaged from such scrap pieces.

A triangular piece is first viewed in such a manner that the base and altitude are suggestive of the X and Y axes, and the vertex of the right angle falling at the origin. The other vertices represent the intercepts. At this point the teacher should review the geometric properties of a square. The pupil is then ready to write the equations of the intersecting lines. The diagonal of the desired square is described by the equation $y=x$. The equation representing the sloping edge of the triangular piece is

$$y=\frac{-ax}{b}+a.$$

The pupil then uses the method of elimination by substitution to solve the pair of equations. The value of x as determined from the solution of the pair of equations is a coordinate of the point of

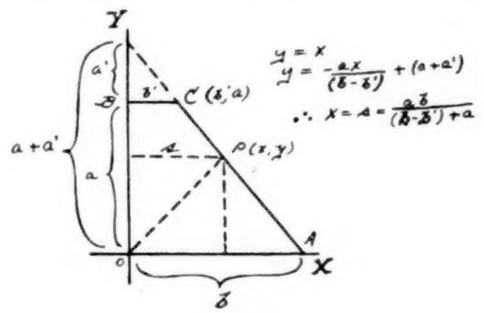


Fig. 2

intersection of the two lines, and also represents the dimension of the square of maximum size. The class then checks the newly found expression by making scale drawings on graph paper. In the final stage of the checking process a bevel square, common square or T-square, and measuring scale should be applied directly to the several objects being measured. The class is now appreciative of the fact that the equation provides for any desired degree of precision, whereas the measurements with square and scale are limited by the human factor as well as the tool.

This problem can be approached from the standpoint of proportionality among the corresponding parts of similar triangles. This is an excellent problem for pupils of the eighth, ninth, and tenth year classes in mathematics.

The figure (Fig. 1) can be the basis for such discussions as: "Radius of Action"; So called "Work Problems"; Addition of fractions whose numerators are unity; Electrical resistances of parallel circuits; Formula for conjugate foci; and many others. The pupil has an opportunity to see that algebra is a powerful tool and not just a plaything.

The problem involving the trapezoid piece requires a little more ingenuity in getting the equation of the sloping edge. The teacher has an excellent opportunity to develop or even review the method of writing the equation of a straight line by the two point and point intercept methods. The pupil can check Figure 2, in a manner similar to the first pattern, and is always quite anxious to apply his findings if he is given the opportunity.

The Solution of Simultaneous Linear Equations

By Nelson B. Conkwright
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J. D. Heide
United States Rubber Co., Passaic, New Jersey

1. *Introduction.* The solution of simultaneous linear equations is frequently a cumbersome task, and many schemes have been devised for facilitating the computation. In the procedure to be described in this paper, the computation is limited to the evaluation of determinants of order two. The process developed is believed to offer several advantages not found in other methods.

The procedure which will be described is essentially the elimination of successive unknowns from the given system of equations, the elimination being effected by the method taught in elementary algebra. The work is accomplished by use of detached coefficients.

In section 6 a rather effective method for finding the rank of a matrix will be described.

2. *Description of the formal procedure in a special case.* In order to solve the simultaneous equations

$$(1) \qquad \sum_{t=1}^{n} a_{st}x_t = k_s, \qquad (s=1,2,3,\cdots,n),$$

we first set up the $n \times (n+1)$ matrix

$$(2) \qquad \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} & k_1 \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} & k_2 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} & k_n \end{pmatrix}.$$

Then form the $(n-1) \times n$ *reduced matrix*

$$(3) \qquad \begin{pmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1,n-1} & k_1' \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2,n-1} & k_2' \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & \cdots & b_{n-1,n-1} & k_{n-1}' \end{pmatrix}$$

in which

$$b_{ij} = \begin{vmatrix} a_{11} & a_{1,j+1} \\ a_{i+1,1} & a_{i+1,j+1} \end{vmatrix},$$

$$k_i' = \begin{vmatrix} a_{11} & k_1 \\ a_{i+1,1} & k_{i+1} \end{vmatrix}.$$

(It will be noted that the process by which the second matrix was obtained is similar to the evaluation of a determinant by the method of pivotal elements.) Now reduce (3) in the same way, and continue to repeat the procedure until the reduction has been carried out $(n-1)$ times. It will be assumed for the present that no one of the matrices thus obtained has a zero element in the upper left-hand corner. This assumption insures that the system (1) will have a unique solution, but we do not propose to dwell upon that point here, since theoretical questions will be considered later.

Let the matrix resulting from the $(n-1)$th reduction be

$$(g_{11} \quad k^{(n-1)}).$$

Then x_n can be found by solving the equation

$$g_{11}x_n = k_1^{(n-1)}.$$

Let the penultimate reduced matrix be

$$\begin{pmatrix} f_{11} & f_{12} & k_1^{(n-2)} \\ f_{21} & f_{22} & k_2^{(n-2)} \end{pmatrix}.$$

Then, x_n being known, the value of x_{n-1} can be determined by means of the equation

$$(4) \qquad f_{11}x_{n-1} + f_{12}x_n = k_1^{(n-2)}.$$

Now that x_n and x_{n-1} are known, we can find x_{n-2} by means of an equation analo-

gous to (4) obtained from the $(n-3)$th reduced matrix. By continuing in this way, all the unknowns are ultimately determined.

In setting up the equations by which the successive unknowns are found, it is not necessary to employ the first row in each of the corresponding matrices. In each instance any row not having zero as its first element may be used. One naturally chooses that row which entails the least computation.

3. *An example.* Let it be required to solve the simultaneous system

$$\begin{aligned} x+y-2z+w&=2\\ 3x-y-z+w&=0\\ 9x+3y+3z-w&=0\\ 2x-3y-3z+2w&=-2. \end{aligned}$$

The initial matrix is

$$\begin{pmatrix} 1 & 1 & -2 & 1 & 2\\ 3 & -1 & -1 & 1 & 0\\ 9 & 3 & 3 & -1 & 0\\ 2 & -3 & -3 & 2 & -2 \end{pmatrix},$$

and the successive reduced matrices are

$$\begin{pmatrix} -4 & 5 & -2 & -6\\ -6 & 21 & -10 & -18\\ -5 & 1 & 0 & -6 \end{pmatrix}$$

$$\begin{pmatrix} -54 & 28 & 36\\ 21 & -10 & -6 \end{pmatrix}$$

$$(-48 \quad -432).$$

Then

$$\begin{aligned} -48w&=-432, & w&=9,\\ 21z-10w&=-\ 6, & z&=4,\\ -5y+z&=-6, & y&=2\\ x+y-2z+w&=2, & x&=-1. \end{aligned}$$

4. *Theory and practice in the general case.* By means of a few obvious changes, the method which has been described can be used to test any system of m simultaneous linear equations in n unknowns for consistency, and to find the solutions if they exist. In practice, the work is simple and straightforward, and precepts are scarcely necessary. The actual procedure is, in fact, much easier than might be inferred from a casual reading of the following exposition. The theory is elementary, but in order to discuss the theoretical details a certain amount of notation is necessary.

Let the system of equations to be investigated be

$$(5)\quad \sum_{j=1}^{n} a_{ij}x_j=k_i, \quad (i=1,2,3,\cdots,m).$$

The augmented matrix of the system (5) will be denoted by A_1, and the successive reduced matrices constructed as in section 2 will be denoted by A_2, A_3, A_4, etc. If in any one of these matrices, say A_s, the leading element is zero, we bring a non-zero element into leading position (if possible) by interchanging rows, and then construct A_{s+1}. If the first column of A_s is composed entirely of zeros, we use the first of the subsequent columns which has a non-zero element. In future references to a reduced matrix A_s, it is to be understood that in this matrix the first column not composed entirely of zeros has its first element different from zero, this condition having been attained, if necessary, by interchange of rows. Any column other than the last column which is composed entirely of zeros will for convenience be omitted in writing the next matrix.

A point will be reached at which no more of these matrices can be constructed in the manner described. Let the final matrix be denoted by A_r. Then A_r will be a row matrix or a column matrix (either of which may be composed entirely of zeros), or else A_r will be a matrix which has all its elements zero except possibly one or more of those in the last column. These are the only possibilities.

Now let the system of equations of which A_s is the augmented matrix be called the system W_s $(s=1, 2, 3, \cdots, r)$. (If A_r is a column matrix, there is no cor-

responding system of equations W_r.) And let $g_s=0$ be the first equation of the system W_s with the constant term transposed to the first member. The system W_s is merely the system which is obtained from (5) by eliminating $x_1, x_2, \cdots, x_{s-1}$ in succession as is done in elementary algebra.

The system composed of the equations of any system W_s and the equations $g_t=0$, $(t=1, 2, \cdots, s-1)$ will be called the system T_s. Any system T_s and the system (5) are equivalent.*

To facilitate the discussion, let the last of the systems W_s be denoted simply by W. Thus W is the system W_{r-1} if A_r is a column matrix. Otherwise W is the system W_r. Let T be the system T_{r-1} or T_r according as A_r is or is not a column matrix. It has already been noted that T and (5) are equivalent. But from the form of T it is clear that T is consistent if and only if the system W is consistent.

In any specific numerical case it is easy to tell by inspection whether the system W is consistent. It is equally easy to find solutions of W if they exist.

An examination of the various possibilities will reveal that W is inconsistent if and only if A_r is composed entirely of zeros except for the last column which contains at least one element different from zero. (There may be only one column.)

It follows that W is consistent if and only if A_r is either a zero matrix or a row matrix whose rank is unity after the last element has been deleted. If W is consistent and A_r is a row matrix, we can solve the equation $g_r=0$ by assigning values to all but one of the unknowns. The remaining unknowns in (5) can then be found as explained in section 2. It can easily be seen that if any A_s has a column (other than the last column) composed entirely of zeros, the value of the corresponding unknown may be assigned at pleasure.

* The reason for this is quite elementary, and can easily be supplied by the reader. See MacDuffee, *Carus Mathematical Monograph*, Number 7, p. 7.

If A_r is a zero matrix, all the equations of W_{r-1} are multiples of the first, and we can obtain a solution of the system W_{r-1} by assigning values to all but one of the unknowns in $g_{r-1}=0$. The solution of (5) can then be completed as before.

All solutions of (5) can obviously be found in the manner described.

5. *Comments.* By using facts pointed out in section 4 the reader will find it easy to justify the interchange of rows in any one of the matrices A_s, and the use of an equation other than $g_s=0$ in the system W_s for the determination of x_s.

It is also easy to show that in any matrix A_s any row may be multiplied by a constant different from zero, and that the rows may be combined in the manner which is customary in working with determinants. In this way it may be possible to simplify the computation outlined in section 4.

If A_1 is symmetric, each of the reduced matrices is also symmetric. In this case only the elements on one side of the principal diagonal of each A_s need to be computed.

In equations (5) the quantities $k_1, k_2, \cdots, k_m$ may all be zero. Thus the discussion in section 4 includes the system of homogeneous equations as a special case.

6. *Determination of the rank of a matrix.* By means of previous results one can deduce a convenient test for the rank of a matrix. Consider any system of homogeneous equations, say

$$(6) \qquad \sum_{j=1}^{n} a_{ij}x_j=0, \qquad (i=1, 2, 3, \ldots, m),$$

and let the coefficient matrix be denoted by A_1. It is known from the standard theory of simultaneous equations that all solutions of (6) can be found by assigning values to a certain $n-q$ of the unknowns and then solving for the remaining q unknowns (whose values are uniquely determined by the $n-q$ assigned values). It is

also known that there is but one such value of q, and that q is the rank of A_1.

Now let A_h be the last of the reduced matrices which has a row not composed entirely of zeros. (Clearly, $h=r-1$ or $h=r$.) Then consideration of earlier results in this paper leads to the conclusion that all the solutions of (6) can be obtained by assigning values to $n-h$ of the unknowns and solving for the others (which are uniquely determined by the $n-h$ assigned values). It follows that $h=q$. Therefore *the rank of any matrix A_1 is equal to q if and only if the last of the reduced matrices to have a row not composed entirely of zeros is A_q.*

In order to simplify the computation, it is permissible to subject any reduced matrix A_s to transformations under which the rank is invariant, and to use the matrix so obtained for the determination of A_{s+1}.

THE ART OF TEACHING

How to Teach the Work Problem

By William R. Lueck
Illinois State Normal University, Normal, Illinois

In the writer's classes in beginning algebra there have been many pupils who experienced difficulty with the so-called "work problem." Because of this fact he busied himself with the task of devising a more satisfactory technique of presenting such problems when first taught. In these times of record machine production, improvement in the presentation of problems involving work assumes added significance. The technique given in this discussion has given good to excellent results in the writer's classes and is presented here for the reader's approval or modification.

To render more clear what is meant by the work problem, let us consider these examples.

1. *John can mow a lawn in two hours. Sam, who is younger and uses a smaller mower, requires five hours to mow the same lawn. In how many hours will the boys finish the lawn if they work together?*
2. *A tank can be filled by one pipe in four hours, and emptied by another in five hours. If both pipes are opened, how long will it take to fill the tank?*
3. *Mr. Clark can plow a field in twelve days when working alone. When assisted by his neighbor, the two together can plow the field in eight days. How long will it take the neighbor when working alone?*
4. *James can build a play house in eight days. After working three days he is joined by Henry, and the two together complete the work in two days. How long will it take Henry alone to do the work?*

Difficulties Encountered

A study of these problems reveals four difficulties.

1. The problems are essentially new to the pupils. They have not, as a group, experienced situations in which such problems have arisen. Hence, their responses are futile attempts at averaging and guessing at the answer.

2. The thinking required also is in contrast to that required of the pupil prior to this time. The *more* time it takes to do a piece of work, the *less* is done in any unit of time. To bring this inverse thinking into clear relief for the class, the teacher confronts one of the slower pupils with this problem. "If it takes you two hours to do some chores and your friend three hours, how long will it take the two of you working together?" Very frequently the response is "five." The class usually realizes the absurdity of the answer without much reflection.

3. Fundamental to solving work problems is the thinking in terms of reciprocals. The unit is the amount of work done in a given interval of time. To think in terms of this unit is distinctly new to the pupil. This is accomplished only by careful teaching.

4. Thinking in terms of reciprocals must be extended and continued until the equation has been developed. When the preliminary knowledge and methods of attack are well understood by the pupils, the development of the equation is greatly simplified.

Presenting the Problems to the Class

In all teaching it is important that the pupils see rather clearly what is expected of them. The class must understand what the new work is in order to actively attack it. As a beginning, the teacher may write on the board the four problems with which this chapter began. Problems written on the board are more potent in securing the attention of the class than those in the textbook.

The class is directed to read the problems carefully in the hope of suggesting possible solutions. There usually will be some comments, and these should be evaluated by the class. They serve to bring the problem into clear focus and develop readiness for the work to come. When not clear, the vocabulary used in stating problems should receive attention at this point. When these preliminary gestures have received attention, the teacher is ready to guide the pupils into the new type of attack required in solving work problems. The class is addressed in the following manner:

"As you know, problems in algebra are solved by first writing an equation. This is true also of the problems at the board. The equation is always based on the facts stated in the problem. Sometimes you will have to derive or find new facts from those given in the problem. You not only read the lines, but also "between them." Very often you will have to think about the facts stated in the problem in a way that is quite new to you. This is all necessary before we can write the equations from which we get the answers to problems.

"Whenever a piece of work is done, it will help us if we know what part of it can be done in one hour, one day, or in any other unit of time. I am going to ask you some simple questions about doing work. At the same time I shall write your answers in a table at the board. This will help us to think about work problems."

(The teacher asks these orally)	(The teacher's work at the board)		
Questions	Total Time Required	Part Done in 1 Unit of Time	Part Done in N Units of Time
1. (To a girl) If it takes you 2 hours to wash the dishes, what part can you do in 1 hour?	2	1/2	
2. (To a boy) You can mow a lawn in 3 hours. What part can you mow in 1 hour?	3	1/3	
3. Suppose you can wash the windows in your house in 5 hours. What part can you do in 1 hour? In 3 hours?	5	1/5	$3\times1/5=3/5$ in 3 hours
4. If a farmer can plow a field in 9 days, what fraction of the work can he do in 1 day? 4 days?	9	1/9	$4\times1/9=4/9$ in 4 days
5. If it takes you m minutes to read a short story, what fraction of it can you read in 1 minute? In 6 minutes?	m	$1/m$	$6\times1/m=6/m$ in 6 minutes
6. Suppose you can walk to school in x minutes. What fraction of the distance do you walk in 1 minute? In 5 minutes?	x	$1/x$	$5\times1/x=5/x$ in 5 minutes
7. If John and Sam together can mow a lawn in h hours, what fraction can they mow in 1 hour? In 3 hours?	h	$1/h$	$3\times1/h=3/h$ in 3 hours

The questions in the table above are directed at individuals in the class. The teacher calls on the more capable pupils first. Ultimately even the slower pupils should be able to give the correct answers to the general situations. Between questions the teacher will often find it necessary to make further explanations. The number of questions that a teacher will direct to the class varies with the ability of the class.

It is better to have too many questions than too few. If the pupils are unable to answer correctly, they must be confronted

with more simple situations until no further errors are forthcoming. The teacher should note that in questions 1 and 2, the word "part" is used instead of the word "fraction." This is as it should be. The word "part" is more easily understood by the less capable pupils than is the word "fraction." As the work progresses, the word "fraction" may be used. It is not enough, however, for a pupil to know what fraction of a given piece of work can be done in one unit of time, knowing the total time required. He must also know how to represent the work that can be done in several units of time. It should be noted that the development of this second concept is delayed for a time, as the table shows. Ultimately the pupils should be thoroughly familiar with both of these ideas.

Writing the Equation

The class is now ready to begin writing the equation for Problem 1. The initial representations are put on the blackboard, with the help of the class, thus:

Let h = the number of hours required by both boys to mow the lawn.
$1/h$ = part of the work both can do in 1 hour.
$1/2$ = part of the work done by John in 1 hour.
$1/5$ = part of the work done by Sam in 1 hour.

If the last three steps are not readily forthcoming, the appropriate part of the previous table should be reviewed. The teacher now pauses in order to develop the fundamental idea on which the equation is to be based, namely, that the fraction of the work done by John in one hour *added* to the fraction done by Sam in one hour equals the fraction of the work done by both boys in one hour.

This fact is so commonplace that many teachers neglect to mention it. They apparently assume that pupils should be capable of applying this idea regardless of where it occurs. However, when a pupil attempts to solve a problem in algebra, he is not looking for everyday facts. His mind is set to discover something difficult, vague, or otherwise incomprehensible from the problem. Hence, the teacher must assist the pupil in finding the simple central idea in the problem on which the equation is to be based.

It is important for the class to see that the two fractions are *added* if the boys are cooperating in the work. Addition is the only way in which this "working together" can be expressed. The equation is then written and solved in the usual manner.

At this point the teacher should direct the class to solve a problem similar in type to the one just solved. Immediate practice of what has been learned is essential to adequate retention.

We have said that when two or more forces are acting cooperatively in doing work, the parts done by each in a given unit of time are added in forming the equation. Now let us find out when to subtract. The attention of the class is now focused on the second problem. In solving it, the class will be assisted by a diagram at the blackboard drawn by the teacher. The pupils will write the initial representations up to the point where they must write the equation. Here the teacher stops the class and a discussion follows. The purpose of this is to make clear that here are two forces acting against each other, and it is the net result that is important in writing the equation. This result is obtained by *subtracting* the amount of work done by the pipe emptying the tank. The result is equal to the net amount of work done by both pipes in one hour. The class should mention other instances of opposing forces. When the central idea on which the equation is based has become clear to the class, the equation for Problem 2 is written and solved.

The class should now be ready to attempt the solution of Problem 3. This represents a slight variation from Problem 1. Most members of the class will readily find the solution. Those who need help should have their thinking guided by means of questions by the teacher, thus:

1. What part of the field can Mr. Clark plow in one day?
2. In eight days Mr. Clark does what fraction of the work?
3. How long does it take both men to plow the field?
4. What fraction of the work is done by both in one day?
5. If it takes the neighbor d days to plow the field, what fraction of it can he do in one day? In 8 days?
6. What is there about the problem that helps in forming the equation? Etc.

The teacher must make sure that the answer to the last question is understood by every pupil. It is often better to discuss this question first as this sets up a goal toward which the pupil may work. Frequently, this procedure tells the pupil what must be represented before the equation can be written. The quantities represented and the facts in the problem needed in writing the equation should be carefully written on paper, especially by the slower pupils.

In Problem 4, a new idea for forming the equation is introduced. The class discussion of the problem should also reveal several new facts. We note first of all that James does a part of the work while working alone, he does another part while working with Henry, and Henry also does a part of the work. The sum of these three parts (or fractions) must equal the whole task. If we represent the three parts by such fractions as 3/8, 2/8, etc. (and we do), then the whole task must be represented by 8/8, 4/4, etc., or 1. In any event, *the pupil must see that the sum of the parts equals the whole.*

The separate parts of the work are now represented, thus:

8 = the number of days required by James alone.
$1/8$ = part of the work done by James in one day.
$3/8$ = part of the work done by James in three days.
$2/8$ = part of the work done by James while working with Henry.
x = the number of days required by Henry alone.
$1/x$ = the part of the work done by Henry in one day.
$2/x$ = the part of the work done by Henry in two days.

Hence: $3/8+2/8+2/x=1$.

The equation is solved and interpreted as before.

The Assignment

The pupils should now have attained a sufficient degree of understanding of the work problem to continue on their own initiative. There will be other slight variations in problems; but if the foregoing work has been thoroughly done, the pupils should be able to surmount these difficulties without much assistance.

The assignment of the problems to be solved by the pupils should be undertaken at once. Care must be exercised in order that a variety of different problem situations occurs in the assignment. The pupil should study the answers obtained to determine if they are reasonable. In Problem 1, it should be pointed out that the answer should be less than two hours, since John *alone* can do the work in that time. The minimum answer to Problem 2 will make a worthwhile discussion. The solution of a dozen or more problems by the class will usually insure adequate practice on this type of problem.

Maintenance

Thought patterns developed in problem solving are subject to deterioration with time in much the same way as the more mechanical phases of algebra. The instructor should check the retention of the pupils not more than two weeks after the initial presentation of the work problems. This can be done by directing to the pupils several questions so framed as to force them to think in terms of reciprocals as found in work problems. The review is concluded by having the class write the equation to a problem not previously solved.

EDITORIAL

Post-War Mathematics

THERE is no doubt but that now is the time for all those who are interested in the improvement of mathematical education to be alert and active. This applies all along the line from the kindergarten on up through the elementary school, the secondary school, and the college.

The task of reorganizing mathematics in the schools and colleges in this country, as has been previously stated in this column, is one in which everybody concerned should cooperate in order that the best solution may be found.

It is gratifying to us to find that others are in accord with our idea of cooperation. The following letter, written recently by the president of The Mathematical Association of America to the officers of the Association and reprinted in full here with his permission should be of great interest to the readers of THE MATHEMATICS TEACHER. Moreover it may help to induce those who should be interested and who occupy strategic positions in the educational world to take appropriate action.

THE MATHEMATICAL ASSOCIATION OF AMERICA

North Hall, University of Wisconsin,
Madison 6, Wisconsin
February 1, 1945.

To the Officers of the Mathematical Association of America.

DEAR COLLEAGUES:

The newspapers and magazines are full of plans for the drastic revision of education in the post-war world. There are plans to increase the offerings of mathematics in the secondary schools, to make its study compulsory, to eliminate it entirely. Some of the proposed reforms are constructive, some are destructive. The situation is fraught with danger as well as with opportunity. It is a situation demanding the leadership of those persons and organizations who really know the answers and who can turn this fluid situation into victory for sound education.

The Mathematical Association of America is, we believe, the foremost American organization devoted primarily to the interests of collegiate mathematics. Collegiate mathematics includes the training of teachers, and is fundamentally affected by the type of training which college students have received in preparatory school. It is impossible to separate the high school problem from the college problem.

The purpose of this letter to the general and sectional officers of the Association is to ask for your help and advice in formulating the policy of the Association with respect to these matters. Do you believe that the Association should, through the action of its committees, keep in close touch with developments in the educational world and take an active part in moulding opinion? If you believe in this policy, please be frank with suggestions as to procedure and with information regarding projects in your part of the nation in which the Association might be helpful.

The problems facing mathematics are but little different from the problems which face the other sciences, or from those which face all of the branches of learning. It therefore seems that the various organizations which are devoted to the interests of true scholarship should make common cause and attempt to work out plans which are in essential agreement with one another. Such plans, backed by the combined influence of a number of scholarly organizations, will have to be given serious consideration in any post-war planning.

One of the problems which appears to be directly the concern of the Association is the maintenance of adequate college preparatory courses in the high schools, and the consequent training of teachers. There is a powerful movement at present in the direction of vocational training in the high schools at the expense of the classical courses. Many educators believe that the majority of high school students can profit more from this type of training than from the traditional courses if there is no prospect of a subsequent collegiate training. This seems highly reasonable.

But the Mathematical Association might well also consider the interests of those more gifted students, small though their number be, who want and can profit by a sound college preparatory type of training. It seems so obvious and trite to say that it is the talented student whose education is essential in order that civilization and culture may endure. The culture of every nation is in the hands of relatively few persons, and the perpetuation of this superior group is the perpetuation of civilization. The necessity of maintaining a select group of highly educated men has long been recognized and has been the objective of European education. Any nation which, in pursuit of some other objective, ignores this one, is committing cultural suicide. But the European means of attaining this end,

namely by educating the sons of the aristocratic and wealthy, is a practice which is not only repugnant to American ideals but is highly wasteful of talent. Any system of education which is worthy to be called American will provide adequate education for the gifted boy or girl irrespective of the social or financial stratum into which he happens to be born.

What the ultimate solution of this problem will be we do not now know, but in some way it will have to be found. If we can contribute anything toward its solution, we should do so. The Junior College with classes starting with the eleventh grade may be an answer. If so, we should be prepared to assume leadership in mapping courses in mathematics for such an institution.

Let us consider another specific instance of the type of reform which might well receive our attention. Engineering schools find it practically necessary to teach physics to freshmen. Physics cannot be adequately taught without the concepts of the differential and integral calculus. Calculus cannot be taught without the rudiments of analytic geometry as a background. Analytic geometry is ordinarily not taught until the freshman or even sophomore year in college. The result is that the freshman physics course is very unsatisfactory from the standpoints both of the student and of the teacher. Analytic geometry could easily be taught in the high school. It is taught in the better preparatory schools, and in some of the large city high schools, for instance in New York and in Cincinnati. We could suggest that it be taught in the twelfth grade of all schools which prepare students for college.

These items are merely thrown out by way of suggestion. What do you think we should stand for, and what means do you think we can employ to bring our ideas to realization?

Sincerely yours,

C. C. MacDuffee, *President*

We feel confident that any suggestions or comments that our readers may feel inclined to make will be gratefully received by President MacDuffee. It also seems possible that the "Commission on Post War Plans" in Mathematics of which Professor Schorling is Chairman would be glad to have similar comments. The second report of Professor Schorling's Commission will appear in the May issue of The Mathematics Teacher.

W. D. R.

New Film Catalog Lists Nearly 700 Visual Aids

Nearly 700 motion pictures and filmstrips produced by the U. S. Government for training and educational purposes are now available for purchase by schools, industry, and other civilian groups, it was announced today by the U. S. Office of Education, Federal Security Agency.

All of these visual aids to education are described in a new catalog which gives synopses, running time, subject groupings, and prices of the various films and filmstrips.

Subjects covered are aviation, agriculture, engineering, machine shop, nursing, shipbuilding, supervision, and many others. They range from "nuts and bolts" films, such as "Cutting a Dovetail Taper Slide," to problems in human relations such as "Employing Blind Workers in Industry," and cover such different subjects as "Celestial Navigation" and "Saving the Garden Crop."

Of the 693 visual aids now available, 440 were produced by the U. S. Office of Education as part of the Visual Aids for War Training program. Another 195 films and filmstrips, produced for military training, have been released for civilian training by the Army and Navy. The U. S. Department of Agriculture produced 58 films.

Copies of the new catalog, just off the press, may be obtained on request from the Division of Visual Aids, U. S. Office of Education, Washington 25, D. C., or from Castle Films, Inc., 30 Rockefeller Plaza, New York 20, N. Y.

IN OTHER PERIODICALS

By Nathan Lazar
Midwood High School, Brooklyn 10, New York

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2. Wilson, E. B., "Note on the *t*-Test," pp. 563–566.
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4. Ott, E. R., "Difference Equations in Average Value Problems," pp. 570–578.
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2. Funkenbusch, William, and Eagle, Edwin, Hyper-Special Tit-Tat-Toe or Tit-Tat-Toe in Four Dimensions," pp. 119–122.
3. Kelly, L. M., "Covering Problems," pp. 123–130.
4. Moorman, R. H., "The Influence of Mathematics on the Philosophy of Leibnitz," pp. 131–140.
5. Court, N. A., "Elements at Infinity in Projective Geometry," pp. 141–146.
6. Schaaf, W. L., "Post-War Planning for Mathematical Education," pp. 147–149.
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3. Carnahan, Walter H., "Selling Mathematics," pp. 148–153.
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NEWS NOTES

The Hastings College Alumni Association is giving special honor this spring to Mr. R. M. McDill, professor of mathematics, who is completing his fiftieth year of teaching.

Mr. McDill's department has been one of the most popular at Hastings College during his twenty-five years on the campus. Of the 1700 alumni of Hastings College more than 200 have majored in mathematics with Professor McDill, who has at all times been a source of admiration and inspiration to his students. Many of his majors have gone into scientific work, where they have made enviable records in their chosen field.

Mr. McDill received his A.B. degree from Indiana University in 1894, his M.A. degree in the same institution in 1898, and did further graduate work at the University of Colorado. He began teaching in New Castle Indiana High School in 1895 and in 1907 he became Professor of Mathematics at the Nebraska Normal School, Fremont, where he served until 1919.

He has written many newspaper and magazine articles including "Exercises Introductory to Geometry," "The Religious Ideas of Scientific Men," and "The Early Education of the Negro in America with Emphasis on the Work of the McKee School."

Mr. McDill has been an Elder in the First Presbyterian Church for many years and is a member of the Kiwanis Club of Hastings. He is a fellow in the American Association for the Advancement of Science and is a member of the following organizations: The Mathematical Association of America, The National Council of Teachers of Mathematics, The Nebraska State Education Association, The Nebraska Academy of Sciences, and the Schoolmasters Club. In 1938 he was chairman of the Nebraska Chapter of the Mathematical Association of America, an organization for which he prepared a number of papers. His name is included in a number of educational biographies including "American Men of Science."

How High Is Up? Talent Search Finalists Know

"If a bullet is shot vertically to an altitude of 675 yards, assuming that it requires as long for its accent for its descent, about how long is the bullet in flight?"

A question of that calibre might cause even agile-witted oldsters to fill wastebaskets with doodle-scarred "figuring paper." But most of the 40 'teen-aged finalists in the fourth annual Science Talent Search—11 of them girls—regarded this and 149 similarly tough questions as so much grist for their mental mills. The correct answer, as application of the laws of acceleration show, is 22½ seconds.

The 40 science-minded finalists, all of them seniors this year in high schools in 15 states, qualified as delegates to the fourth annual Science Talent Institute—climax of the Science Talent Search—partly on the basis of a difficult science aptitude examination, from which the above question was taken.

1945 Summer Session at Teachers College, Columbia University in the Teaching of Mathematics

Teachers College, Columbia University, will offer the following courses in the teaching of mathematics in the summer session of 1945 which begins on July 2 and ends on August 10:

By Professor John R. Clark: Teaching arithmetic in the elementary school; Teaching algebra in secondary schools. By Dr. Nathan Lazar: Teaching geometry in secondary schools; History of mathematics. By Mr. G. R. Mirick: Elementary mechanics (statics); Observation and participation in the teaching of geometry. By Professor W. D. Reeve: Teaching and supervision of mathematics: junior high school; Teaching and supervision of mathematics: senior high school. By Professor W. S. Schlauch: Business mathematics. By Professor C. N. Shuster: Field work in mathematics Preflight training in Civil Aeronautics Navigation (July 2 to 20); Navigation (July 23 to August 10).

There will be held during the summer session, on consecutive Thursdays beginning on July 5, five special lectures and discussions in which all of the instructors above and other persons from outside will participate for the purpose of bringing before the students vital questions relating to the reorganization and teaching of mathematics in the post-war world. There will be opportunity for discussion in which all of the students will be invited to take part. These conferences have come to be a common meeting place for all students, instructors, and guests, and thus serve both professional and social ends. Registration for these special lectures and discussions is not necessary and all those who are interested in the improvement of mathematical education are invited to attend.

The Rhode Island Mathematics Teachers' Association held its Annual Spring Meeting at Brown University in Providence on Saturday March 17, at 2 P.M. Professor Carl N. Shuster of the State Teachers College in Trenton, N. J. spoke on the topic "Field Work and Other Applications of Secondary Mathematics."

M. L. Herman of the Moses Brown School is president and Harriet C. Whitaker of Lincoln School is secretary-treasurer of the Association.

The Women's Mathematics Club of Chicago and Vicinity held its Annual Joint Meeting with the Men's Club on Friday March 16, at Huyler's. Dr. Mayme I. Logsdon, Associate Professor of Mathematics at the University of Chicago and author of "A Mathematician Explains" spoke on the topic "What Shall We Do Now?"

President, Miss Marion Eckel, Kelly High School; *Vice-President*, Miss Marie Graff, South Shore High School; *Secretary*, Miss Virginia Terhune, Proviso Twp. High School; *Treasurer*, Miss Edith Levin, Englewood High School; *Program Chairman*, Mrs. Elsie P. Johnson, Oak Park High School.